ASYMPTOTIC ANALYSIS FOR SCATTERING OFF ONE-DIMENSIONAL COMPACTLY SUPPORTED WEAK POTENTIALS

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Abstract. Scattering theory, especially in quantum mechanics and wave propagation, provides profound insights into the interactions within various physical systems. In this paper, we focus on the mathematical characterization and asymptotic analysis of the scattering matrix under mixed and Neumann boundary conditions within the context of one-dimensional compactly supported potential ODEs.

1. Introduction and preliminaries

We study the asymptotic behavior of the scattering matrix, $S(\lambda)$, for a one-dimensional second-order non-linear ordinary differential equation under mixed and Neumann boundary conditions. Assuming the potential q has compact support, we derive the complete asymptotic expansion of $S(\lambda)$ for the mixed-boundary condition with $\nu \neq 0$ and the leading term for the Neumann boundary condition ($\nu = 0$). Our analysis reveals distinct instability in the behavior of the scattering matrix: under Neumann condition, $S(\lambda) \overset{\lambda \to 0}{\to} 1$, contrasting with a convergence to -1 under other conditions.

The paper is structured as follows: In Sec. 2, we formally present our settings and few basic definitions. In Sec. 3, we derive the complete asymptotic expansion of S-matrix under mixed-boundary condition with $\nu \neq 0$ and the leading term in the asymptotic expansion under Nuemann boundary condition. Sec. 4 focuses on the justification of such expansion and Appendix A gives a brief review of Picard-Lindelöf Theorem.

Notations. Let $\Omega \subseteq \mathbb{C}$ be a non-empty open subset of the set of complex numbers $\mathbb{C}$. For any complex number $z = x + \mathrm{i}y$, its complex conjugate is denoted by z^*, where $z^* = x - \mathrm{i}y$. The set of real numbers is denoted by $\mathbb{R}$ and the set of positive real numbers is denoted by $\mathbb{R}_{>0}$. Additionally, we use $\mathbb{N} := \{1, 2, 3, \ldots\}$ to denote the set of natural numbers.

2. Abstract problem formulation

Definition 2.1 (Continously differentiable functions)**.** Let $k \in \{0\} \cup \mathbb{N}$. A function $f : \Omega \to \mathbb{C}$ is *k times continuously differentiable* if all partial derivatives of f up to and including order k exist and are continuous on Ω.

We use the following notations for the standard function classes:

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- $\mathrm{C}^0(\Omega) = \mathrm{C}(\Omega)$ is the set of continuous functions on Ω;
- $\mathrm{C}^k(\Omega) := \{f : \Omega \to \mathbb{C} \mid f \text{ is } k \text{ times continuously differentiable}\}$;
- $\mathrm{C}^\infty(\Omega) := \bigcap_{k=0}^{\infty} \mathrm{C}^k(\Omega)$ is the set of infinitely differentiable functions (or *smooth* functions) on Ω.

Definition 2.2 (Support of continuous functions)**.** Let $f \in \mathrm{C}(\Omega)$. The *support* of f is the closure of the set $\{x \in \Omega, f(x) \neq 0\}$ relative to Ω:

$$\operatorname{supp}(f) := \Omega \cap \overline{\{x \in \Omega \mid f(x) \neq 0\}}.$$

Definition 2.3 (Test functions)**.** $\mathcal{D}(\Omega) \subseteq \mathrm{C}^\infty(\Omega)$ is the set of smooth functions with compact support:

$$\mathcal{D}(\Omega) := \{\varphi \in \mathrm{C}^\infty(\Omega) : \operatorname{supp}(\varphi) \text{ is compact}\}.$$

We often call this set the class of *test functions*.

The standard types of boundary conditions for equations defined on $\mathbb{R}_{>0}$ with dependent variable $u(x)$ that we will consider are

- **Dirichlet boundary condition:** $u(0) = 0$;
- **Neumann boundary condition:** $u'(0) = 0$;
- **Mixed-boundary condition:** $u'(0) = \nu u(0)$ for $\nu \in \mathbb{R}$.

Remark 2.4. Note that both Dirichlet boundary condition and Neumann boundary condition are special cases of mixed-boundary condition, where the former can be viewed as $u(0) = \frac{u'(0)}{\nu}$ with $\nu \to \infty$ and the latter can be viewed as the case when $\nu = 0$.

Boundary-value problem (BVP) with mixed-boundary data. Consider the following one-dimensional BVP: find $u(x) \in \mathrm{C}^2(\mathbb{R}_+)$ such that

$$\begin{cases} u''(x) + \underbrace{\lambda^2(q(x)+1)}_{=:Q(x)} u(x) = 0, \\ u'(0) = \nu u(0), \end{cases} \tag{2.1}$$

where $\nu \in \mathbb{R}$, $\lambda > 0, q \in \mathcal{D}(\mathbb{R}), q \geq 0$, and $\operatorname{supp}(q) \subseteq [0, R]$ for some $R > 0$. Clearly, we have $Q(x) = \lambda^2$ for $x > R$.

Proposition 2.5. *There exists a unique solution $\eta \in \mathrm{C}^2(\mathbb{R}_+)$ to the BVP (2.1) that additionally satisfies*

$$\eta(x) = e^{i\lambda x} + S e^{-i\lambda x}, \quad x > R, \tag{2.2}$$

for some $S = S_\lambda \in \mathbb{C}$ with $|S| = 1$.

Remark 2.6. S is often called the **scattering matrix** or **S-matrix**. One well-known property of the scattering matrix is that it is unitary, namely, $SS^* = 1$. In our one-dimensional case, S is a simply a complex scalar of modulus one.

Proof. We first prove this proposition for $\nu = 0$. By the global version of the Picard-Lindelöf theorem A.4, there exists a unique solution $y \in \mathrm{C}^2([0, K])$ for any $K \in \mathbb{R}$ with the boundary-values $y'(0) = 0$ and $y(0) = 1$. To show η in (2.2) exists and is unique, let us consider $\eta = My$ for some $M \in \mathbb{C}$ we will choose appropriately later. Observe that $\forall x \in \mathbb{R}_+, \eta(x)$ satisfies $\eta''(x) + Q(x)\eta(x) = 0$ and $\eta'(0) = My'(0) = 0$. Thus, η is also a solution to the BVP (2.1) with $\nu = 0$. For $x > R$, we have

$$y''(x) + \lambda^2 y(x) = 0 \implies y(x) = Ae^{i\lambda x} + Be^{-i\lambda x},$$

where $A, B \in \mathbb{C}$ depend on λ. We claim that $A \neq 0$. Suppose, for the sake of contradiction that, $A = 0$. Then, by construction,

$$y(x) = Be^{-i\lambda x}, \quad x > R \tag{2.3}$$

satisfies

$$\begin{cases} y''(x) + Q(x)y(x) &= 0, \\ y'(0) &= 0, \\ y(0) &= 1. \end{cases}$$

Observe that for a real constant $T > R$, we obtain that

$$\begin{aligned} 0 &= \int_0^T \left(y''(x) + Q(x)y(x)\right) y^*(x)dx \\ &= \int_0^T y(x)(y''^*(x) + Q(x)y^*(x))dx + [y'(x)y^*(x) - y(x)y'^*(x)]_0^T \\ &= y'(T)y(T)^* - y(T)y'(T)^* \\ &= -B\lambda ie^{-i\lambda T}B^*e^{i\lambda T} - Be^{-i\lambda T}B^*\lambda e^{i\lambda T} \\ &= -2i\lambda|B|^2 \overset{\lambda>0}{\implies} B = 0, \end{aligned}$$

where the second equality is by using integration by parts twice. Thus, (2.3) becomes $y(x) = 0$ for $x > R$. By the uniqueness of solution y, we have $y(x) = 0$ for $x > 0$ which contradicts the fact that $y(0) = 1$. Now, if we choose $M = \frac{1}{A}$, then

$$\eta(x) = e^{i\lambda x} + \frac{B}{A}e^{-i\lambda x}, \quad x > R$$

where $S_\lambda := \frac{B(\lambda)}{A(\lambda)}$. It remains to show that $SS^* = 1$. We use the same technique here. Observe that for a real constant $T > R$, we obtain that

$$\begin{aligned} 0 &= \int_0^T \left(y''(x) + Q(x)y(x)\right) y^*(x)dx \\ &= \int_0^T y(x)(y''^*(x) + Q(x)y^*(x))dx + [y'(x)y^*(x) - y(x)y'^*(x)]_0^T \\ &= y'(T)y(T)^* - y(T)y'(T)^* \\ &= i\lambda\left(1 + S_\lambda e^{2i\lambda T} - S_\lambda e^{-2i\lambda T} - |S_\lambda|^2\right) + i\lambda\left(1 + S_\lambda e^{-2i\lambda T} - S_\lambda e^{2i\lambda T} - |S_\lambda|^2\right) \\ &= 2i\lambda(1 - |S_\lambda|^2) \overset{\lambda>0}{\implies} |S_\lambda| = 1, \end{aligned}$$

which completes the proof for the case where $\nu = 0$. Let us now consider the case for $\nu \neq 0$. Again by the global version of the Picard-Lindelöf theorem A.4, there exists a unique solution $y \in \mathrm{C}^2([0, K])$ for any $K \in \mathbb{R}$ but this time we use the boundary-values $y'(0) = 1$ and $y(0) = \nu^{-1}$. Consider $\eta = My$ for some

$M \in \mathbb{C}$ we will choose appropriately later. Observe that $\forall x \in \mathbb{R}_+, \eta(x)$ satisfies $\eta''(x) + Q(x)\eta(x) = 0$ and $\eta'(0) = My'(0) = M = \nu\eta(0)$. Thus, η is also a solution to the BVP (2.1) with $\nu \neq 0$. Since $\nu \in \mathbb{R}$, one can observe that

$$\begin{aligned}
[y'(x)y^*(x) - y(x)y'^*(x)]_0^T &= y'(T)y(T)^* - y(T)y'(T)^* + y(0)y'(0)^* - y'(0)y(0)^* \\
&= y'(T)y(T)^* - y(T)y'(T)^* + \frac{1^*}{\nu} - \frac{1}{\nu^*} \\
&= y'(T)y(T)^* - y(T)y'(T)^*.
\end{aligned}$$

Given this observation, the remainder of the proof proceeds identically to the case when $\nu = 0$. □

3. Asymptotic expansion of S-matrix under mixed-boundary condition

In this section, we derive the full asymptotic expansion of the scattering matrix S under the mixed-boundary condition. We first consider the case when $\nu \neq 0$ and later we will consider the Neumann case ($\nu = 0$) separately.

Suppose that η given by (2.2) admits an asymptotic expansion in terms of λ for $x > 0$ of the form [1]

$$\eta_\lambda(x) = \sum_{n=0}^{\infty} \eta_n(x)\lambda^n, \quad x > 0. \tag{3.1}$$

Suppose that the scattering matrix S given by (2.2) also admits an asymptotic expansion in terms of λ of the form

$$S_\lambda = \sum_{n=0}^{\infty} S_n\lambda^n. \tag{3.2}$$

For convenience, we omit the subscript λ when the parameter of our asymptotic expansion is unambiguous.

Due to (3.2) and the Taylor expansion of the exponential function, it follows immediately that for $x > R$

$$\begin{aligned}
\eta(x) &= e^{i\lambda x} + Se^{-i\lambda x} \\
&= \sum_{n=0}^{\infty} \frac{(i\lambda x)^n}{n!} + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-ix)^k}{k!} S_{n-k} \right) \lambda^n \\
&= \sum_{n=0}^{\infty} a_n(x)\lambda^n,
\end{aligned}$$

where $a_n(x) := \frac{(ix)^n}{n!} + \sum_{k=0}^{n} \frac{(-ix)^k}{k!} S_{n-k}$. Thus, we obtain another asymptotic expansion of η in terms of λ for $x > R$ of the form

$$\eta(x) = \sum_{n=0}^{\infty} a_n(x)\lambda^n. \tag{3.3}$$

Our goal now is to find formulas for $\{S_n\}_{n=1}^{\infty}$.

[1] See section 4 for details.

3.1. **Asymptotic expansion of S-matrix under mixed-boundary condition with $\nu \neq 0$.** To begin with, substituting (3.1) into (2.1), we obtain the following equations for $n \in \{0,1\}$

$$\begin{cases} \eta_n''(x) = 0, \\ \eta_n'(0) = \nu\eta_n(0). \end{cases} \tag{3.4}$$

Moreover, we obtain the following cascade of equations for η_n for $n \geq 2$

$$\begin{cases} \eta_n''(x) = -(1+q(x))\eta_{n-2}(x), \\ \eta_n'(0) = \nu\eta_n(0), \end{cases} \tag{3.5}$$

where $\nu \in \mathbb{R}\setminus\{0\}$, $\lambda > 0, q \in \mathcal{D}(\mathbb{R}), q \geq 0$, and $\operatorname{supp}(q) \subseteq [0,R]$ for some $R > 0$. To be concrete, let us solve (3.4) and (3.5) with $n = 2$ by matching the asymptotic expansion in (3.3) to see what $\{S_n\}_{n=0}^2$ and $\{\eta_n\}_{n=0}^2$ look like.

$\mathcal{O}(1)$: For $n = 0$, we have

$$\begin{cases} \eta_0''(x) = 0, \\ \eta_0'(0) = \nu\eta_0(0). \end{cases} \tag{3.6}$$

By integrating twice, we find that $\eta_0(x) = A_0x + B_0$ where $A_0, B_0 \in \mathbb{C}$ are some constants. By the boundary condition, we find that $B_0 = A_0\nu^{-1}$ and hence $\eta_0(x) = A_0\left(x+\nu^{-1}\right)$. Observe that $a_0(x) = 1+S_0$ in (3.3). Asymptotic matching, i.e. matching for $x > R$, gives $A_0 = 0$ and $S_0 = -1$. Thus, we have that

$$S_0 = -1 \quad \text{and} \quad \eta_0(x) = 0. \tag{3.7}$$

Remark 3.1. Under the mixed-boundary condition, $\nu \neq 0$, we have $S(\lambda) \overset{\lambda\to 0}{\to} -1$.

$\mathcal{O}(\lambda)$: For $n = 1$, we have

$$\begin{cases} \eta_1''(x) = 0, \\ \eta_1'(0) = \nu\eta_1(0). \end{cases} \tag{3.8}$$

Similarly, we find that $\eta_1(x) = A_1x+B_1$ where $A_1, B_1 \in \mathbb{C}$ are some constants. By the boundary condition, we find that $B_1 = A_1\nu^{-1}$ and hence $\eta_1(x) = A_1\left(x+\nu^{-1}\right)$. Observe that $a_1(x) = i(1-S_0)x+S_1$ in (3.3). Asymptotic matching gives $A_1 = 2i$ and $S_1 = 2i\nu^{-1}$. Thus, we have that

$$S_1 = 2i\nu^{-1} \quad \text{and} \quad \eta_1(x) = 2i\left(x+\nu^{-1}\right). \tag{3.9}$$

$\mathcal{O}(\lambda^2)$: For $n = 2$, we have

$$\begin{cases} \eta_2''(x) = -(1+q(x))\eta_0(x), \\ \eta_2'(0) = \nu\eta_2(0). \end{cases} \tag{3.10}$$

Since we have already found $\eta_0(x) = 0$, we find that $\eta_2(x) = A_2x + B_2$ where $A_2, B_2 \in \mathbb{C}$ are some constants. By the boundary condition, we find that $B_2 = A_2\nu^{-1}$ and hence $\eta_1(x) = A_2\left(x+\nu^{-1}\right)$. Observe that $a_2(x) = -\frac{(1+S_0)}{2}x^2 - iS_1x + S_2$ in (3.3). Asymptotic matching gives $A_2 = -iS_1 = 2\nu^{-1}$ and $S_2 = 2\nu^{-2}$. Thus, we have that

$$S_2 = 2\nu^{-2} \quad \text{and} \quad \eta_2(x) = 2\nu^{-1}\left(x+\nu^{-1}\right). \tag{3.11}$$

Full asymptotic expansion. Let us now turn to the full asymptotic expansion for η and, in particular, S. The next result states that, for every n, the function η_n is a polynomial for $x > R$.

Lemma 3.2. *Let $n \in \mathbb{N} \setminus \{1\}$. Consider the following BVP: find $\eta_n \in \mathrm{C}^2(\mathbb{R}_{>0})$ such that*

$$\begin{cases} \eta_n''(x) = -(1+q(x))\eta_{n-2}(x), \\ \eta_n'(0) = \nu\eta_n(0), \end{cases} \tag{3.12}$$

where $\nu, \lambda,$ and q are as in (3.5). Then there exists a unique polynomial P_n of degree at most n such that $\eta_n(x) = P_n(x) + A_n\left(x + \nu^{-1}\right)$ for $x > R$ where $A_n \in \mathbb{C}$ is some suitable constant.

Proof. Clearly, η_2 satisfies Lemma 3.2 with $P_2 = 0$ and $A_2 = 2\nu^{-1}$. Assume that Lemma 3.2 is true for $n = k$. Let us now prove this result for $n = k+2$. By integrating the ODE in (3.12) twice, we find that

$$\eta_{k+2}(x) = \underbrace{\int_0^x \int_0^t -(1+q(\tau))\eta_k(\tau)d\tau dt}_{=:\tilde{\eta}_{k+2}(x)} + A_{k+2}\left(1+\nu^{-1}\right). \tag{3.13}$$

We aim to show that $\tilde{\eta}_{k+2}(x)$ is a polynomial of degree at most $k+2$ for $x > R$. Splitting $\tilde{\eta}_{k+2}(x)$ into two integrals, we have

$$\tilde{\eta}_{k+2}(x) = \underbrace{\int_0^R \int_0^t -(1+q(\tau))\eta_k(\tau)d\tau dt}_{=:I_1} + \underbrace{\int_R^x \int_0^t -(1+q(\tau))\eta_k(\tau)d\tau dt}_{=:I_2}. \tag{3.14}$$

Observe that, by integration by parts,

$$\begin{aligned} I_1 &= \int_0^R t' \int_0^t -(1+q(\tau))\eta_k(\tau)d\tau dt \\ &= \underbrace{R\int_0^R -(1+q(\tau))\eta_k(\tau)d\tau}_{const} - \underbrace{\int_0^R t\left(-(1+q(\tau))\right)\eta_k(t)dt}_{const}. \end{aligned}$$

Moreover, by our inductive hypothesis, we have

$$I_2 = \int_R^x \left(\underbrace{\int_0^R (-1+q(\tau))\eta_k(\tau)d\tau}_{const} - \int_R^t P_k(x) + A_k\left(x+\nu^{-1}\right)d\tau \right) dt,$$

which is a polynomial of degree at most $k+2$. ☐

Remark 3.3. Note that $\eta_0(x)$ and $\eta_1(x)$ also satisfies Lemma 3.2 with $P_0 \equiv P_1 \equiv 0$.

The importance of Lemma 3.2 is that we can now match the polynomial η_n with a_n, for $x > R$ to determine the A_n and S_n. Indeed, from (3.3) and Lemma

3.2, it follows that for the asymptotic ansatz (3.1)-(3.2) to be valid, one must necessarily have

$$P_n(x) + A_n\left(x + \nu^{-1}\right) = a_n(x) := \frac{(\mathrm{i}x)^n}{n!} + \sum_{k=0}^{n} \frac{(-\mathrm{i}x)^k}{k!} S_{n-k}, \quad x > R. \tag{3.15}$$

Denoting $P_n(x) = \sum\limits_{k=0}^{n} b_k^{(n)} x^k$; then by equating coefficients in (3.15), we arrive at the constraints

$$b_0^{(n)} = S_n - A_n\nu^{-1}, \tag{3.16}$$

$$b_1^{(n)} = -\mathrm{i}S_{n-1} - A_n, \tag{3.17}$$

$$b_j^{(n)} = \delta_{n,j}\frac{\mathrm{i}^n}{n!} + \frac{(-\mathrm{i})^j}{j!} S_{n-j}, \quad 2 \le j \le n, \tag{3.18}$$

From (3.16) and (3.17) it follows that

$$A_n = b_1^{(n)} + \mathrm{i}S_{n-1}$$

and, therefore

$$S_n = b_0^{(n)} - (b_1^{(n)} + iS_{n-1})\nu^{-1}.$$

We shall now represent $b_0^{(n)}$ and $b_1^{(n)}$ in terms of the potential q and A_j with some $j \le n$.

Proposition 3.4. *Denoting $b_0^{(0)} = b_0^{(1)} = b_1^{(1)} = 0$. For $n \in \mathbb{N} \setminus \{1\}$, one has*

$$b_0^{(n)} = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} A_{n-2j} \langle q, tL^{j-1}h \rangle,$$

$$b_1^{(n)} = -\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} A_{n-2j} \langle q, L^{j-1}h \rangle,$$

where $h(t) = t+\nu^{-1}$, $Lu(x) = \int_0^x \int_0^t -(1+q(\tau))u(\tau)d\tau dt$, and $\langle q, f\rangle := \int_0^R q(t)f(t)dt$.

Proof. We have $\eta_n(x) = P_n(x)+A_n h(x)$, where $P_n(x) = \int_0^x \int_0^t -((1 + q(\tau))\,\eta_{n-2}(\tau)d\tau dt$. Consider the linear map L defined as

$$Lu(x) := \int_0^x \int_0^t -((1 + q(\tau))\, u(\tau)d\tau dt.$$

By linearity, we have $LP_0 = 0$ since $P_0 = 0$. We also have $P_2 = L\eta_0 = L(P_0 + A_0h) = A_0Lh$ and $P_4 = L\eta_2 = L(P_2 + A_2h) = A_0L^2h + A_2Lh$. By induction, we have for $k \in \{0\} \cup \mathbb{N}$

$$P_{2k}(x) = \sum_{j=1}^{k} A_{2(k-j)} L^j h.$$

Similarly, we have

$$P_{2k+1}(x) = \sum_{j=1}^{k} A_{2(k-j)+1} L^j h.$$

Observe that

$$\begin{aligned}
L(L^{j-1}h) &= \int_0^x \int_0^t -\left((1+q(\tau)\right) L^{j-1}h(\tau)d\tau dt \\
&= \int_0^R t' \int_0^t -\left((1+q(\tau)\right) L^{j-1}h(\tau)d\tau dt + \int_R^x t' \int_0^t -\left((1+q(\tau)\right) L^{j-1}h(\tau)d\tau dt \\
&= \left[t\int_0^t -\left((1+q(\tau)\right) L^{j-1}h(\tau)d\tau\right]_{t=0}^R - \int_0^R tQ(t)L^{j-1}h(t)dt \\
&+ \left[t\int_0^t -\left((1+q(\tau)\right) L^{j-1}h(\tau)d\tau\right]_{t=R}^x - \int_R^x t[-\left((1+q(\tau)\right)]L^{j-1}h(t)dt \\
&= x\int_0^x -\left((1+q(\tau)\right) L^{j-1}h(\tau)dr - \int_0^R t[-\left((1+q(\tau)\right)]L^{j-1}h(t)dt + \underbrace{\int_R^x tL^{j-1}h(t)dt}_{\text{no linear terms}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
b_0^{(2k)} &= \sum_{j=1}^k A_{2(k-j)}\langle q, tL^{j-1}h\rangle, \\
b_1^{(2k)} &= -\sum_{j=1}^k A_{2(k-j)}\langle q, L^{j-1}h\rangle.
\end{aligned}$$

Similarly, one can show that

$$\begin{aligned}
b_0^{(2k+1)} &= \sum_{j=1}^k A_{2(k-j)+1}\langle q, tL^{j-1}h\rangle, \\
b_1^{(2k+1)} &= -\sum_{j=1}^k A_{2(k-j)+1}\langle q, L^{j-1}h\rangle.
\end{aligned}$$

Here $\langle q, f\rangle := \int_0^R q(t)f(t)dt$. $\square$

Summarising the above results, determines all terms in the asymptotic expansions (3.1)-(3.2):

Corollary 3.5. *The full asymptotic expansion η and of the scattering matrix S under the mixed-boundary condition with $\nu \neq 0$ are given by:*

$$\begin{aligned}
&A_0 = 0, && A_1 = 2i, && A_n = -iS_{n-1} - b_1^{(n)} \textit{ for } n \geq 2; \\
&S_0 = -1, && S_1 = 2i\nu^{-1}, && S_n = b_0^{(n)} - (b_1^{(n)} + iS_{n-1})\nu^{-1} \textit{ for } n \geq 2,
\end{aligned}$$

where

$$b_0^{(n)} = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} A_{n-2j} \langle q, tL^{j-1}h \rangle,$$

$$b_1^{(n)} = -\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} A_{n-2j} \langle q, L^{j-1}h \rangle,$$

$h(t) = t + \nu^{-1}$, $Lu(x) = \int_0^x \int_0^t -(1+q(\tau))u(\tau)d\tau dt$, *and* $\langle q, f \rangle := \int_0^R q(t)f(t)dt$.

For consistency, we need to demonstrate that A_n and S_n given in Corollary 3.5, satisfy (3.18). Put another way, for these A_n and S_n, we need to demonstrate that, for each and every n, η_n equals a_n in the matching region $x > R$:

Proposition 3.6. *Suppose* $\{A_j\}_{j\in\mathbb{N}}$ *and* $\{S\}_{j\in\mathbb{N}}$ *are given by Corollary 3.5. Then*

$$\eta_n(x) = \frac{(\mathrm{i}x)^n}{n!} + \sum_{k=0}^{n} \frac{(-\mathrm{i}x)^k}{k!} S_{n-k}, \qquad x > R, \quad n \in \mathbb{N}.$$

Proof. We need to establish (3.16)-(3.18). However, equations (3.16) and (3.17) follow from definition of A_n and S_n. Let us show that (3.18) holds.

First note, from (3.12), the fact $q \equiv 0$ and the fact $\eta_n = P_n + A_n(x + \nu^{-1})$ for $x > R$, one has

$$P_n''(x) = -P_{n-2}(x) - A_{n-2}\left(x + \nu^{-1}\right), \quad x > R,$$

or in component form

$$j(j-1)b_j^{(n)} = -b_{j-2}^{(n-2)} - \delta_{j,3}A_{n-2} - \delta_{j,2}A_{n-2}\nu^{-1}, \quad 2 \le j \le n. \tag{3.19}$$

We shall prove the desired result by mathematical induction.

(1) For $n = 2$. Notice that, since $S_0 = -1$, the right-hand-side of (3.18), for $n = 2$, reads

$$\frac{\mathrm{i}^2}{2!} + \frac{(-i)^2}{2!} S_0 = 0.$$

So, we need to show $b_2^{(2)} = 0$. From (3.19) it follows that

$$2b_2^{(2)} = -b_0^{(0)} - A_0\nu^{-1}.$$

Now, since $A_0 = b_0^{(0)} = 0$, it follows that $b_2^{(2)} = 0$. Therefore, (3.18) holds for $n = 2$.

(2) Suppose (3.18) holds for $n - 1 \ge 2$. We shall show it holds for n. Fix $2 \le j \le n$. Equation (3.16) and (3.17) give

$$b_0^{(n-2)} = S_{n-2} - A_{n-2}\nu^{-1}, \quad b_1^{(n-2)} = -\mathrm{i}S_{n-3} - A_{n-2}.$$

Substituting this into (3.19) gives

$$j(j-1)b_j^{(n)} = (\delta_{j,3} + \delta_{j,2} - 1)b_{j-2}^{(n-2)} + \delta_{j,3}\mathrm{i}S_{n-3} - \delta_{j,2}S_{n-2}. \tag{3.20}$$

Equation (3.18), for $n - 2$ gives

$$b_{j-2}^{(n-2)} = -\delta_{n-2,j-2}\frac{\mathrm{i}^n}{(n-2)!} - \frac{(-\mathrm{i})^j}{(j-2)!} S_{n-j}. \tag{3.21}$$

Substituting this into (3.20), and the fact $n-2>1$ gives

$$j(j-1)b_j^{(n)} = (\delta_{j,3}+\delta_{j,2}-1)\big(-\delta_{n-2,j-2}\frac{\mathrm{i}^n}{(n-2)!} - \frac{(-\mathrm{i})^j}{(j-2)!}S_{n-j}\big) + \delta_{j,3}\mathrm{i}S_{n-3} - \delta_{j,2}S_{n-2}$$
$$= \delta_{n-2,j-2}\frac{\mathrm{i}^n}{(n-2)!} + \frac{(-\mathrm{i})^j}{(j-2)!}S_{n-j}.$$

Dividing both sides of the above equation by $j(j-1)$ gives (3.18).

□

3.2. **Asymptotic expansion of S-matrix under Neumann boundary condition.** Under the Neumann boundary condition, substituting (3.1) into (2.1) gives the following equations for $n \in \{0,1\}$

$$\begin{cases} \eta_n''(x) = 0, \\ \eta_n'(0) = 0. \end{cases} \tag{3.22}$$

Moreover, we obtain the following cascade of equations for η_n for $n \geq 2$

$$\begin{cases} \eta_n''(x) = -(1+q(x))\eta_{n-2}(x), \\ \eta_n'(0) = 0, \end{cases} \tag{3.23}$$

where $\nu \in \mathbb{R}\setminus\{0\}$, $\lambda > 0, q \in \mathcal{D}(\mathbb{R}), q \geq 0$, and $\operatorname{supp}(q) \subseteq [0,R]$ for some $R>0$. To be concrete, let us solve (3.22) by matching the asymptotic expansion in (3.3) to see what $\{S_n\}_{n=0}^1$ and $\{\eta_n\}_{n=0}^1$ look like.
$\mathcal{O}(1)$:. For $n=0$, we have

$$\begin{cases} \eta_0''(x) = 0, \\ \eta_0'(0) = 0. \end{cases} \tag{3.24}$$

By integrating twice, we find that $\eta_0(x) = A_0x + B_0$ where $A_0, B_0 \in \mathbb{C}$ are some constants. By the boundary condition, we find that $A_0 = 0$ and hence $\eta_0(x) = B_0$. Observe that $a_0(x) = 1 + S_0$ in (3.3). Asymptotic matching gives $S_0 = B_0 - 1$ so we need to go one step further to find S_0.
$\mathcal{O}(\lambda)$:. For $n=1$, we have

$$\begin{cases} \eta_1''(x) = 0, \\ \eta_1'(0) = 0. \end{cases} \tag{3.25}$$

Similarly, we find that $\eta_1(x) = A_1x + B_1$ where $A_1, B_1 \in \mathbb{C}$ are some constants. By the boundary condition, we find that $A_1 = 0$ and hence $\eta_1(x) = B_1$. Observe that $a_1(x) = i(1-S_0)x + S_1$ in (3.3). Asymptotic matching gives $B_1 = i(1-S_0)x + S_1$. Thus, we have that

$$S_0 = 1 \quad \text{and} \quad \eta_0(x) = 2. \tag{3.26}$$

Remark 3.7. Under Neumann boundary condition, we have $S(\lambda) \overset{\lambda\to 0}{\to} 1$.

4. Justification of the asymptotic expansion of S-matrix

Here, we shall justify the formal asymptotic expansion for the scattering matrix S. In particular, we shall prove the following result:

Theorem 4.1. *Fix $\beta > 0$ and $N \in \mathbb{N}$. Then, there exists $\lambda_0 > 0$ and $C > 0$ such that*

$$\left| S - \sum_{n=0}^{N} \lambda^n S_n \right| \leq C\lambda^{N+1-2\beta} |\ln \lambda|^{N-1} \qquad \text{for all } |\lambda| < \lambda_0.$$

Proof of Theorem 4.1. Let $\delta > 0$ be such that $\lambda |\ln \lambda| < 1$. Consider $|\lambda| < \lambda_1 := \min\{1, e^{-R}, e^{-2}, \delta\}$. Notice, for such λ that $|\ln \lambda| > 2$ and $|\ln \lambda| > R$.

We begin with some auxiliary results that will be utilised in the proof of Theorem 4.1. Fix $N \in \mathbb{N}$, set

$$\eta_{\text{in}} := \sum_{n=0}^{N} \lambda^n \eta_n$$

and

$$\eta_{\text{out}} := e^{\mathrm{i}\lambda x} + \left(\sum_{n=0}^{N} \lambda^n S_n \right) e^{-\mathrm{i}\lambda x}$$

where η_n and S_n are given by Lemma 3.2 and Corollary 3.5. Let $\chi \in C_0^\infty(0, \infty)$ be a smooth cut off function (bounded by one) that is equals one on $(0, 1)$ and zero on $(2, \infty)$. Let us consider the approximation

$$\eta_{app}(x) := \chi\big(\tfrac{x}{|\ln \lambda|}\big)\eta_{\text{in}}(x) + \Big(1 - \chi\big(\tfrac{x}{|\ln \lambda|}\big)\Big)\eta_{\text{out}}(x).$$

Proposition 4.2. *For $N \in \mathbb{N}$, $\beta > 0$. and $|\lambda| < \lambda_1$, one has*

$$\eta_{\text{in}} = \eta_{\text{out}} + r \quad\ \in (|\ln \lambda|, 2|\ln \lambda|).$$

Furthermore,

$$\left(\int_{|\ln \lambda|}^{2|\ln \lambda|} e^{2\beta x} |r(x)|^2 \,\mathrm{d}x \right)^{1/2} \leq C_0 \lambda^{N+1-2\beta} |\ln \lambda|^{N+1}$$

and

$$\left(\int_{|\ln \lambda|}^{2|\ln \lambda|} e^{2\beta x} |r'(x)|^2 \,\mathrm{d}x \right)^{1/2} \leq C_0 \lambda^{N+1-2\beta} |\ln \lambda|^{N}$$

for some constant $C_0 > 0$ independent of λ.

Proof. We shall use the fact

$$\int_{|\ln \lambda|}^{2|\ln \lambda|} e^{2\beta x} x^{2n} \,\mathrm{d}x \leq \frac{1}{2\beta} \lambda^{-4\beta} (2|\ln \lambda|)^{2n}, \tag{4.1}$$

for $\beta > 0$. Indeed, for $0 < a < b$ and $n \in \mathbb{N}$, by integration by parts

$$\begin{aligned}
\int_a^b e^{2\beta x} x^{2n}\,\mathrm{d}x &= \frac{1}{2\beta} e^{2\beta x} x^{2n}|_{x=a}^b \underbrace{- \frac{n}{\beta}\int_a^b e^{2\beta x} x^{2n-1}\,\mathrm{d}x}_{\leq 0} \\
&\leq \frac{1}{2\beta} e^{2\beta x} x^{2n}|_{x=a}^b = \frac{1}{2\beta} e^{2\beta b} b^{2n} \underbrace{- \frac{1}{2\beta} e^{2\beta a} a^{2n}}_{\leq 0} \\
&\leq \frac{1}{2\beta} e^{2\beta b} b^{2n}.
\end{aligned}$$

Now, by Proposition 3.6, one has

$$\eta_{\mathrm{in}}(x) = \sum_{n=0}^{N} \lambda^n \left(\frac{(\mathrm{i}x)^n}{n!} + \sum_{k=0}^{n} \frac{(-\mathrm{i}x)^k}{k!} S_{n-k} \right).$$

Moreover, since

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} \lambda^n S_n\right) e^{-\mathrm{i}\lambda x} &= \left(\sum_{n=0}^{\infty} \lambda^n S_n\right)\left(\sum_{n=0}^{\infty} \lambda^n \frac{(-\mathrm{i}x)^n}{n!}\right) \\
&= \sum_{n=0}^{N} \lambda^n \left(\sum_{k=0}^{n} \frac{(\mathrm{i}x)^k}{k!} S_{n-k}\right) + \sum_{n=0}^{N} S_n \left(\sum_{j=N+1}^{\infty} \lambda^j \frac{(-\mathrm{i}x)^{j-n}}{(j-n)!}\right),
\end{aligned}$$

then

$$\eta_{\mathrm{out}}(x) = \eta_{\mathrm{in}}(x) + r(x); \quad r(x) := \sum_{n=N+1}^{\infty} \lambda^n \frac{(\mathrm{i}x)^n}{n!} + \sum_{n=0}^{N} S_n \left(\sum_{j=N+1}^{\infty} \lambda^j \frac{(-\mathrm{i}x)^{j-n}}{(j-n)!}\right). \tag{4.2}$$

$$\|r\| := \left(\int_{|\ln\lambda|}^{2|\ln\lambda|} e^{2\beta x} |r(x)|^2\,\mathrm{d}x\right)^{1/2},$$

one has,

$$\|r\| \leq \sum_{n=N+1}^{\infty} \frac{\lambda^n}{n!} \left(\int_{|\ln\lambda|}^{2|\ln\lambda|} e^{2\beta x} x^{2n}\,\mathrm{d}x\right)^{1/2} + \sum_{n=0}^{N} |S_n| \sum_{j=N+1}^{\infty} \frac{\lambda^j}{(j-n)!} \left(\int_{|\ln\lambda|}^{2|\ln\lambda|} e^{2\beta x} x^{2(j-n)}\,\mathrm{d}x\right)^{1/2}.$$

Now, (4.1) gives

$$\begin{aligned}
\sum_{n=N+1}^{\infty} \frac{\lambda^n}{n!} \left(\int_{|\ln\lambda|}^{2|\ln\lambda|} e^{2\beta x} x^{2n} \, \mathrm{d}x \right)^{1/2} &\leq \sum_{n=N+1}^{\infty} \frac{\lambda^n}{n!} \frac{1}{\sqrt{2\beta}} \lambda^{-2\beta} (2|\ln\lambda|)^n \\
&= \frac{\lambda^{-2\beta}}{\sqrt{2\beta}} \sum_{n=N+1}^{\infty} \frac{\lambda^n}{n!} (2|\ln\lambda|)^n \\
&\leq \frac{\lambda^{-2\beta}}{\sqrt{2\beta}} \lambda^{N+1} (|\ln\lambda|)^{N+1} \underbrace{\sum_{n=N+1}^{\infty} \frac{2^n}{n!}}_{\leq e^2} \\
&\leq \frac{e^2}{\sqrt{2\beta}} \lambda^{N+1-2\beta} (|\ln\lambda|)^{N+1}.
\end{aligned}$$

The second to last inequality comes from the fact $\lambda|\ln\lambda| < 1$. Similarly, for $0 \leq n \leq N$, one has

$$\begin{aligned}
\sum_{j=N+1}^{\infty} \frac{\lambda^j}{(j-n)!} \left(\int_{|\ln\lambda|}^{2|\ln\lambda|} e^{2\beta x} x^{2(j-n)} \, \mathrm{d}x \right)^{1/2} &\leq \frac{\lambda^{-2\beta}}{\sqrt{2\beta}} |\ln\lambda|^{-n} \sum_{j=N+1}^{\infty} \lambda^j |\ln\lambda|^j \frac{2^{j-n}}{(j-n)!} \\
&\leq \frac{e^2}{\sqrt{2\beta}} \lambda^{N+1-2\beta} |\ln\lambda|^{N+1} |\ln\lambda|^{-n},
\end{aligned}$$

and, so

$$\begin{aligned}
\sum_{n=0}^{N} |S_n| \sum_{j=N+1}^{\infty} \frac{\lambda^j}{(j-n)!} \left(\int_{|\ln\lambda|}^{2|\ln\lambda|} e^{2\beta x} x^{2(j-n)} \, \mathrm{d}x \right)^{1/2} &\leq \frac{e^2}{\sqrt{2\beta}} \lambda^{N+1-2\beta} |\ln\lambda|^{N+1} \underbrace{\sum_{n=0}^{N} |S_n| |\ln\lambda|^{-n}}_{\leq \|S\|_N \frac{1}{1-|\ln\lambda|^{-1}} \leq 2\|S\|_N} \\
&\leq \frac{2e^2 \|S\|_N}{\sqrt{2\beta}} \lambda^{N+1-2\beta} |\ln\lambda|^{N+1},
\end{aligned}$$

for $\|S\|_N := \max_{n=0,\ldots,N} \{S_n\}$ and $|\ln\lambda| \geq 2$. Putting all this together gives

$$\|r\| \leq \frac{e^2}{\sqrt{2\beta}} \left(1 + 2\|S\|_N\right) \lambda^{N+1-2\beta} |\ln\lambda|^{N+1}.$$

To calculate the norm of r' we proceed exactly as above. Differentiating (4.2) gives

$$\eta'_{\mathrm{out}}(x) = \eta'_{\mathrm{in}}(x) + r'(x); \quad r'(x) = \sum_{n=N+1}^{\infty} \mathrm{i}^n \lambda^n \frac{x^{n-1}}{(n-1)!} + \sum_{n=0}^{N} S_n \left(\sum_{j=N+1}^{\infty} (-\mathrm{i})^{j-n} \lambda^j \frac{x^{j-n-1}}{(j-n-1)!} \right).$$

Then, we readily compute

$$\|r'\| \leq \frac{e^2}{\sqrt{2\beta}} \big(1 + 2\|S\|_N\big) \lambda^{N+1-2\beta} |\ln\lambda|^N.$$

$\square$

For the proof of Theorem 4.1 we shall introduce the following function spaces:

$$W^l_\beta := \{u \,:\, e^{\beta x}u \in H^l(0,\infty)\}$$

for whole number l and real number β. Note that W^l_β is a Hilbert space when equipped with the inner product that induces the norm

$$\|u\|_{W^l_\beta} := \|e^{\beta x}u\|_{H^l(0,\infty)}.$$

Fix $\lambda \in \mathbb{R}$. Let us consider the following problem: for given $(g, f) \in \mathbb{C} \times W^0_\beta$ find $(c, v) \in \mathbb{C} \times W^2_\beta$ such that $u = ce^{-\mathrm{i}\lambda x} + v$ satisfies

$$\begin{cases} u'' + \lambda^2(q+1)u = f & \text{in } (0,\infty), \\ u'(0) - \nu u(0) = g. \end{cases} \tag{4.3}$$

The following well-posedness result holds:

Lemma 4.3. *For each $\beta \neq 0$, there exists a $\lambda_2 > 0$ such that for all $|\lambda| < \lambda_2$, problem (4.3) is uniquely solvable. Furthermore, one has*

$$\|(c, v)\|_{\mathbb{C}\times \mathbb{W}^2_\beta} \leq K\|(g, f)\|_{\mathbb{C}\times W^0_\beta}, \tag{4.4}$$

for some positive constant K that is independent of f and g.

To prove Lemma 4.3, we use the following result:

Proposition 4.4. *Let $f \in W^0_\beta$, $\beta \neq 0$, then*

$$v(x) := \int_x^\infty f(t)\,\mathrm{d}t$$

belongs to W^0_β.

Proof. It is sufficient to establish

$$\int_0^\infty e^{2\beta x}\left|\int_x^\infty f(t)\,\mathrm{d}t\right|^2 \mathrm{d}x \leq \frac{1}{\beta^2}\int_0^\infty e^{2\beta x} f^2(x)\,\mathrm{d}x. \tag{4.5}$$

To that end, by Cauchy-Schwarz inequality

$$\left|\int_x^\infty f(t)\,\mathrm{d}t\right|^2 \leq \left(\int_x^\infty e^{\beta t} f^2(t)\,\mathrm{d}t\right)\left(\int_x^\infty e^{-\beta t}\,\mathrm{d}t\right)$$

and

$$\int_x^\infty e^{-\beta t}\,\mathrm{d}t = \frac{1}{\beta}e^{-\beta x},$$

plus

$$\int_x^\infty e^{\beta t} f^2(t)\,\mathrm{d}t = \int_0^\infty \chi_{[x,\infty)}(t) e^{\beta t} f^2(t)\,\mathrm{d}t$$

So,

$$\left|\int_x^\infty f(t)\,\mathrm{d}t\right|^2 \leq \frac{1}{\beta}e^{-\beta x}\int_0^\infty \chi_{[x,\infty)}(t) e^{\beta t} f^2(t)\,\mathrm{d}t.$$

Then, multiplying both sides of the inequality by $e^{2\beta x}$ and integrating with respect to x gives

$$\int_0^\infty e^{2\beta x}\left|\int_x^\infty f(t)\,\mathrm{d}t\right|^2\,\mathrm{d}x \le \frac{1}{\beta}\int_0^\infty e^{\beta x}\int_0^\infty \chi_{[x,\infty)}(t)e^{\beta t}f^2(t)\,\mathrm{d}t\mathrm{d}x. \tag{4.6}$$

By applying Fubini's theorem to the RHS gives

$$\begin{aligned}\int_0^\infty e^{\beta x}\int_0^\infty \chi_{[x,\infty)}(t)e^{\beta t}f^2(t)\,\mathrm{d}t\mathrm{d}x &= \int_0^\infty\int_0^\infty e^{\beta x}\chi_{[x,\infty)}(t)e^{\beta t}f^2(t)\,\mathrm{d}t\mathrm{d}x\\ &= \int_0^\infty\int_0^\infty e^{\beta x}\chi_{[x,\infty)}(t)e^{\beta t}f^2(t)\,\mathrm{d}x\mathrm{d}t\\ &= \int_0^\infty e^{\beta t}f^2(t)\int_0^\infty \chi_{[x,\infty)}(t)e^{\beta x}\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Now,

$$\int_0^\infty \chi_{[x,\infty)}(t)e^{\beta x}\,\mathrm{d}x = \int_0^t e^{\beta x}\,\mathrm{d}x = \frac{1}{\beta}(e^{\beta t}-1).$$

Thus

$$\begin{aligned}\int_0^\infty e^{\beta x}\int_0^\infty \chi_{[x,\infty)}(t)e^{\beta t}f^2(t)\,\mathrm{d}t\mathrm{d}x &= \int_0^\infty \frac{1}{\beta}(e^{\beta t}-1)e^{\beta t}f^2(t)\,\mathrm{d}t\\ &= \frac{1}{\beta}\int_0^\infty \left(e^{2\beta t}f^2(t)-e^{\beta t}f^2(t)\right)\mathrm{d}t\\ &= \frac{1}{\beta}\int_0^\infty e^{2\beta t}f^2(t)\,\mathrm{d}t - \frac{1}{\beta}\int_0^\infty e^{\beta t}f^2(t)\,\mathrm{d}t\\ &\le \frac{1}{\beta}\int_0^\infty e^{2\beta t}f^2(t)\,\mathrm{d}t.\end{aligned}$$

The last inequality follows from the fact that $\frac{1}{\beta}\int_0^\infty e^{\beta t}f^2(t)\,\mathrm{d}t \ge 0$. Combining this with (4.6) gives

$$\int_0^\infty e^{2\beta x}\left|\int_x^\infty f(t)\,\mathrm{d}t\right|^2\,\mathrm{d}x \le \frac{1}{\beta^2}\int_0^\infty e^{2\beta t}f^2(t)\,\mathrm{d}t;$$

i.e. (4.5) holds. □

Proof of Lemma 4.3. Let $\mathbb{L}_\lambda : \mathbb{C}\times W^2_\beta \to \mathbb{C}\times W^0_\beta$ be given by the mapping

$$\mathbb{L}_\lambda(c,v) = \Big((ce^{-\mathrm{i}\lambda x}+v)'(0) - \nu(ce^{-\mathrm{i}\lambda x}+v)(0), (ce^{-\mathrm{i}\lambda x}+v)'' + \lambda^2(1+q)(ce^{-\mathrm{i}\lambda x}+v)\Big).$$

Since q is bounded then clearly $\mathbb{L}_\lambda$ is a continuous operator. To prove the Lemma it is equivalent to prove that $\mathbb{L}_\lambda$ has a bounded inverse for small enough λ.

Step 1. Here we consider the case $\lambda = 0$. We shall prove $\mathbb{L}_0$ is boundedly invertible. Fix $(g,f)\in\mathbb{C}\times W^0_\beta$. From Proposition 4.4, it follows that

$$v(x) := \int_x^\infty\int_y^\infty f(t)\,\mathrm{d}t\,\mathrm{d}y$$

belongs to W^2_β and solves $v'' = f$ in $(0,\infty)$. Thus

$$(c,v), \qquad \text{for } c = \nu^{-1}v'(0) - v(0) - \nu^{-1}g,$$

satisfies
$$\mathbb{L}_0(c,v)=(g,f);$$
Namely, $\mathbb{L}_0$ is surjective. Furthermore, by construction (cf. (4.5)), it follows that
$$\|(c,v)\|_{\mathbb{C}\times\mathbb{W}^2_\beta}\leq C_1\|(g,f)\|_{\mathbb{C}\times W^0_\beta} \tag{4.7}$$
for some $C_1>0$ independent of f.

It remains to show that $\mathbb{L}_0$ is injective. Suppose (c,v) satisfies $\mathbb{L}_0(c,v)=(0,0)$. Then $v''=0$ and $v'(0)-\nu(c+v(0))=0$; that is
$$v(x)=a(x+\nu^{-1})+c$$
for some constant a. However, v will belong to $\mathbb{W}^2_\beta$ only if $a=c=0$; that is $\mathbb{L}_0$ is injective. Thus $\mathbb{L}_0^{-1}$ exists, and from (4.7) it follows that the inverse is bounded; indeed (4.7) can be rewritten as
$$\|\mathbb{L}_0^{-1}f\|_{\mathbb{C}\times\mathbb{W}^2_\beta}\leq C_1\|(g,f)\|_{\mathbb{C}\times W^0_\beta}\quad \forall g\in\mathbb{C},\forall f\in W^0_\beta. \tag{4.8}$$
Step 2. Here we consider the case $\lambda\neq 0$ but sufficiently small. First note that, one has
$$\mathbb{L}_\lambda(c,v)-\mathbb{L}_0(c,v)=\Big(-\mathrm{i}\lambda c,\lambda^2 qce^{-\mathrm{i}\lambda x}+\lambda^2(1+q)v\Big).$$
Therefore, since $q\in C_0^\infty(0,R)$, for $|\lambda|\leq 1$, one has
$$\|\mathbb{L}_\lambda(c,v)-\mathbb{L}_0(c,v)\|_{\mathbb{W}^0_\beta\times\mathbb{C}}\leq|\lambda|C_2\sqrt{|c|^2+\|v\|^2_{W^0_\beta}} \tag{4.9}$$
for some C_2 independent of c and v. Now,
$$\mathbb{L}_\lambda=\mathbb{L}_0+\mathbb{L}_\lambda-\mathbb{L}_0=\big(1+(\mathbb{L}_\lambda-\mathbb{L}_0)\mathbb{L}_0^{-1}\big)\mathbb{L}_0.$$
Now, consider the operator $T:\mathbb{C}\times W^0_\beta\to\mathbb{C}\times W^0_\beta$ given by $T:=(\mathbb{L}_\lambda-\mathbb{L}_0)\mathbb{L}_0^{-1}$. From, (4.9) and (4.8), one has
$$\|T(g,f)\|_{W^0_\beta}\leq|\lambda|C_2C_1\sqrt{|g|^2+\|f\|^2_{W^0_\beta}}\qquad\forall g\in\mathbb{C},\forall f\in W^0_\beta;$$
consequently, $\|T\|<1$ for $|\lambda|<\lambda_2:=\min\{1,(C_1C_2)^{-1/2}\}$. Thus, T has a convergent Neumann series and in particular $(1-T)^{-1}$ exists and is bounded. Consequently, $\mathbb{L}_\lambda^{-1}$ exists, and is bounded, for $|\lambda|<\lambda_0$ and is given by
$$\mathbb{L}_\lambda^{-1}:=\mathbb{L}_0^{-1}\big(1-(\mathbb{L}_\lambda-\mathbb{L}_0)\mathbb{L}_0^{-1}\big)^{-1}.$$
□

Proof of Theorem 4.1. Fix λ such that Proposition 4.2 and Lemma 4.3 holds; i.e. $|\lambda|<\lambda_0:=\min\{\lambda_1,\lambda_2\}$.

We compute
$$\eta''_{app}+\lambda^2(1+q)\eta_{app}=f_\lambda,$$
for
$$f_\lambda(x)=\frac{1}{|\ln\lambda|^2}\chi''\big(\tfrac{x}{|\ln\lambda|}\big)\left[\eta_{\text{in}}(x)-\eta_{\text{out}}(x)\right]+\frac{2}{|\ln\lambda|}\chi'\big(\tfrac{x}{|\ln\lambda|}\big)\left[\eta'_{\text{in}}(x)-\eta'_{\text{out}}(x)\right].$$
By construction $\chi'\big(\frac{x}{|\ln\lambda|}\big)$ and $\chi''\big(\frac{x}{|\ln\lambda|}\big)$ are non-zero only for $x\in(|\ln\lambda|,2|\ln\lambda|)$. Consequently, from Proposition 4.2, it follows that

$$\| \frac{1}{|\ln\lambda|^2} \chi''\big(\tfrac{x}{|\ln\lambda|}\big) \left[\eta_{\text{in}}(x) - \eta_{\text{out}}(x)\right] \|_{W^0_\beta} \leq C_0 \frac{\lambda^{N+1-2\beta} |\ln\lambda|^{N+1}}{|\ln\lambda|^2},$$

and

$$\| \frac{1}{|\ln\lambda|} \chi'\big(\tfrac{x}{|\ln\lambda|}\big) \left[\eta'_{\text{in}}(x) - \eta'_{\text{out}}(x)\right] \|_{W^0_\beta} \leq C_0 \frac{\lambda^{N+1-2\beta} |\ln\lambda|^{N}}{|\ln\lambda|}.$$

That is,

$$\|f_\lambda\|_{W^0_\beta} \leq 3C_0 \lambda^{N+1-2\beta} |\ln\lambda|^{N-1}. \tag{4.10}$$

Finally, notice that the difference $r := \eta_\lambda - \eta_{app}$ satisfies

$$r = ce^{-\mathrm{i}\lambda x} + v$$

for

$$c = S - \sum_{n=o}^{N} \lambda^n S_n$$

and some $v \in W^2_\beta$. Furthermore, r solves

$$\begin{cases} r'' + \lambda^2 (q+1) r = -f_\lambda & \text{in } (0,\infty), \\ r'(0) - \nu r(0) = 0. \end{cases}$$

That is, $(c, v) \in \mathbb{C} \times W^2_\beta$ solves (4.3) for $(g, f) = (0, -f_\lambda)$; therefore, from Lemma 4.3 (namely (4.4)) and (4.10) it follows that

$$\|(c, v)\|_{\mathbb{C}\times\mathbb{W}} \leq 3KC_0 \lambda^{N+1-2\beta} |\ln\lambda|^{N-1}.$$

In particular, the desired inequality holds, indeed:

$$\left| S - \sum_{n=0}^{N} \lambda^n S_n \right| = |c| \leq 3KC_0 \lambda^{N+1-2\beta} |\ln\lambda|^{N-1}.$$

□

Appendix A. Picard-Lindelöf theorem

Definition A.1 (Contraction mapping)**.** Let (X, d) be a metric space. A function $T : X \to X$ is a *contraction mapping* if $\exists c \in (0, 1)$ such that $\forall x, y \in X$

$$d(T(x), T(y)) \leq c d(x, y).$$

Definition A.2 (Fixed point)**.** Let $T : X \to X$ be a function. A *fixed point* of T is a point $x \in X$ such that $T(x) = x$.

Theorem A.3 (Contraction mapping theorem)**.** *Let (X, d) be a non-empty complete metric space and let $T : X \to X$ be a contraction mapping. Then T has a unique fixed point $x \in X$, which satisfies $x = \lim_{n\to\infty} x_n$, where $x_{n+1} = T(x_n)$ and $x_0 \in X$ is arbitrary.*

Proof. Existence: Let $x_0 \in X$ and define the sequence $\{x_n\}$ recursively by $x_{n+1} = T(x_n)$. Let us first show that $\{x_n\}$ is a Cauchy sequence. Note that

$$d(x_2, x_1) = d(T(x_1), T(x_0)) \leq c d(x_1, x_0),$$
$$d(x_3, x_2) = d(T(x_2), T(x_1)) \leq c d(x_2, x_1) \leq c^2 d(x_1, x_0),$$

and by induction, $\forall n \geq 1, d(x_{n+1}, x_n) \leq c^n d(x_1, x_0)$. Now let $m, n \in \mathbb{N}$ and w.l.o.g. suppose $m > n$. Then

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \ldots + d(x_{n+1}, x_n) \\ &\leq c^{m-1} d(x_1, x_0) + c^{m-2} d(x_1, x_0) + \ldots c^n d(x_1, x_0) \\ &= d(x_1, x_0) \sum_{i=n}^{m-1} c^i \\ &\leq d(x_1, x_0) \sum_{i=n}^{\infty} c^i \overset{n\to\infty}{\to} 0. \end{aligned}$$

Thus, $\{x_n\}$ is indeed Cauchy. Since X is complete, $\exists x \in X$ such that $x_n \overset{n\to\infty}{\to} x$. Let us now show that x is a fixed point of T. Note that $T(x_n) \overset{n\to\infty}{\to} T(x)$ as

$$d(T(x_n), T(x)) \leq c d(x_n, x) \overset{n\to\infty}{\to} 0.$$

Since $T(x_n) = x_{n+1}$, $T(x_n)$ is just a subsequence of $\{x_n\}$ and hence $T(x_n) \overset{n\to\infty}{\to} x$. By the uniqueness of limits, we conclude that $T(x) = x$. *Uniqueness:* Suppose that x, y are two fixed points of T with $x \neq y$. Then

$$d(T(x), T(y)) \leq cd(x, y) \Rightarrow d(x, y) \leq cd(x, y) \Rightarrow 1 \leq c,$$

contradicting the fact that T is a contraction mapping. Thus, the fixed point x is unique. □

Theorem A.4 (Global Version of the Picard-Lindelöf Theorem). *Let $I := [a, b] \subset \mathbb{R}, x_0 \in I$ and $y_0 \in \mathbb{R}^n$, and $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function which satisfies a global Lipschitz condition with respect to y as follows:*

$$\exists L > 0 : \quad \forall x \in I, \quad \forall y_1, y_2 \in \mathbb{R}^n : \quad \|f(x, y_1) - f(x, y_2)\|_2 \leq L \|y_1 - y_2\|_2 .$$

Then the IVP

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0$$

has a unique solution $y \in \mathrm{C}^1(I \to \mathbb{R}^n)$.

Proof. **Transformation to an integral equation:** If $y \in \mathrm{C}^1(I \to \mathbb{R}^n)$ is a solution to the IVP, then the Fundamental Theorem of Calculus states

$$y(x) = y_0 + \int_{s=x_0}^{x} f(t, y(t)) \mathrm{d}s =: (T(y))(x), \quad x \in I.$$

Conversely, if $y \in \mathrm{C}(I \to \mathbb{R}^n)$ is a solution to this integral equation, then (again by the Fundamental Theorem of Calculus), the integral in the right-hand side is a differentiable function of x, hence $y \in \mathrm{C}^1(I \to \mathbb{R}^n)$, and we can differentiate this integral equation to obtain the IVP. Hence, the IVP and this integral equation are equivalent.

Applying the Contraction Mapping Theorem: We choose $U = \mathrm{C}(I \to \mathbb{R}^n)$ with the special norm

$$\|u\|_U := \sup_{x\in I} e^{-(L+1)|x-x_0|}\|u(x)\|_2$$

with L being the constant appearing in the Lipschitz condition. It can be shown that $(U, \|\cdot\|_U)$ is complete and $T : U \to U$ is well defined. It remains to show T is a contraction mapping.

First we note that

$$(T(u) - T(\tilde{u}))(x) = \int_{x_0}^{x} f(t, u(t)) - f(t, \tilde{u}(t))\mathrm{d}t$$

and therefore we find

$$\begin{aligned}
&e^{-(L+1)|x-x_0|}\|(T(u) - T(\tilde{u}))(x)\|_2 = e^{-(L+1)|x-x_0|}\left\|\int_{x_0}^{x} f(t, u(t)) - f(t, \tilde{u}(t))\mathrm{d}t\right\|_2 \\
&\le e^{-(L+1)|x-x_0|}\int_{\min(x,x_0)}^{\max(x,x_0)} \|f(t, u(t)) - f(t, \tilde{u}(t))\|_2 \ \mathrm{d}t \\
&\le e^{-(L+1)|x-x_0|}\int_{\min(x,x_0)}^{\max(x,x_0)} L\|u(t) - \tilde{u}(t)\|_2 \ \mathrm{d}t \\
&\le e^{-(L+1)|x-x_0|}L\int_{\min(x,x_0)}^{\max(x,x_0)} e^{(L+1)|t-x_0|}\underbrace{e^{-(L+1)|t-x_0|}\|u(t) - \tilde{u}(t)\|_2}_{\le\|u-\tilde{u}\|_U} \ \mathrm{d}t \\
&\le \|u - \tilde{u}\|_U e^{-(L+1)|x-x_0|}L\int_{\min(x,x_0)}^{\max(x,x_0)} e^{(L+1)|t-x_0|}\mathrm{d}t \\
&= \|u - \tilde{u}\|_U e^{-(L+1)|x-x_0|}L \cdot \frac{1}{L+1}\left(e^{(L+1)|x-x_0|} - e^0\right) \\
&\le \frac{L}{L+1}\|u - \tilde{u}\|_U.
\end{aligned}$$

This shows that T is a contraction mapping with constant $c = L/(L + 1) < 1$. Therefore, CMT implies that T has exactly one fixed point y in U. □

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