Notes on Graph Theory

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2023

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[0]These notes are based on the *MATH0029 Graph Theory and Combinatorics* module lectured by Dr. *John Talbot* at UCL during Term 1 of the 2023-2024 academic year. However, they have been extensively adapted and are not endorsed by the lecturer. Any mistakes contaitned within these notes are mine almost surely. Should you find any mistakes, please feel free to email me at peter.wang.21@alumni.ucl.ac.uk.

1 Basic counting: sets and tuples

If X is a set then $|X|$ is the **size** or **cardinality** of X. Most of the sets we will encounter in this course will be finite (with the frequent exception of the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$). Our **"canonical" set of size** n (where $n \in \mathbb{N}$) is denoted by $[n] := \{1, 2, \ldots, n\}$.
The **empty set** is denoted $\emptyset$. If A and B are sets, then A is a **subset** of B if every element of A is also an element of B. We denote this by $A \subseteq B$.
If A and B are sets, then their **union** is $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ and their **intersection** is $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. The **set difference** of A and B is $A \backslash B = \{x \mid x \in A \text{ and } x \notin B\}$. The **symmetric difference** of A and B is $A \Delta B = (A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B)$.
If A and B are sets, then their **cartesian product** is $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.
If A and B are sets such that $A \cap B = \emptyset$, then we say A and B are **disjoint**. If we have a union of two disjoint sets, we emphasize this by writing $A \dot{\cup} B$. If $A_1, A_2, \ldots, A_n$ are sets, then we say they are **pairwise disjoint** if $A_i \cap A_j = \emptyset$ for all $1 \leq i < j \leq n$.

Lemma 1.1. *Let A and B be finite sets. Then*

(i) $|A \backslash B| = |A| - |A \cap B|$;

(ii) $|A \cup B| = |A| + |B| - |A \cap B|$;

(iii) $|A \times B| = |A| \cdot |B|$;

(iv) If $A_1, A_2, \ldots, A_n$ are pairwise disjoint sets, then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|;$$

(v) If $A_1, A_2, \ldots, A_n$ are sets, then

$$\left|\bigcup_{i=1}^{n} A_i\right| \leq \sum_{i=1}^{n} |A_i|.$$

Proof. Omitted. □

Remark 1.2. The so-called **sum rule** holds for disjoint sets: if A and B are disjoint sets, then $|A \cup B| = |A| + |B|$. This follows from lemma 1.1.

For any integer $k \geq 1$, we define k **factorial** to be $k! = k(k-1) \cdots 2 \cdot 1$. We define $0! = 1$. For integers $n \geq k \geq 1$, we define the k**-th falling factorial** of n by $(n)_k = n(n-1) \cdots (n-k+1)$. So $(k)_k = k!$.
Let X be a set. A k**-tuple of elements** from X is $(x_1, x_2, \ldots, x_k)$ where $x_i \in X$ for $1 \leq i \leq k$. Note that the elements of a k-tuple need not be distinct. We will denote **the set of** k**-tuples** from X by X^k.
Let X be a set. A k**-tuple of distinct elements** from X is $(x_1, x_2, \ldots, x_k)$ where $x_i \in X$ for $1 \leq i \leq k$ and $x_i \neq x_j$ for $i \neq j$. We will denote **the set of** k**-tuples of distinct elements** from X by $(X)_k$.
If X is a set of size n and $0 \leq k \leq n$, then **the family of** k**-sets** from X is

$$\binom{X}{k} = \{A \subseteq X : |A| = k\}.$$

Let $0 \leq k \leq n$ be integers. Recalling that $0! = 1$ and we define the **binomial coefficient** by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n)_k}{k!}.$$

If X is a set, then the **power set** of X is the family of all subsets of X :

$$\mathscr{P}(X) = \{A \mid A \subseteq X\}.$$

Often we work with subsets of a given "universal set" X. In this case we can define the **complement** of a set $A \subseteq X$ by $A^c = X \backslash A$.

Lemma 1.3. *Let $1 \le k \le n$ be integers and X be a set of size n. Then*

1. *There are n^k different k-tuples of elements from X, so $\left|X^k\right| = n^k$;*
2. *There are $n(n-1)\cdots(n-k+1)$ different k-tuples of distinct elements from X, so $|(X)_k| = (n)_k$;*
3. *There are $n!$ distinct permutations of the elements of X;*
4. *There are $\binom{n}{k}$ distinct k-sets from X, so $\left|\binom{X}{k}\right| = \binom{n}{k}$;*
5. *$\binom{n}{k} = \binom{n}{n-k}$;*
6. *$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$;*
7. *$|\mathscr{P}(X)| = 2^n$.*

Proof. Omitted. □

2 Graphs

2.1 Basic definitions

Definition 2.1 (Graph, vertex, edge, order, size). A **graph** is a pair (V, E) of sets with $E \subseteq \binom{V}{2}$. An element of V is a **vertex** and an element of E is an **edge**. We denote the set of vertices and the set of edges of a graph G by $V(G)$ and $E(G)$ respectively. The **order** of a graph G is the number of vertices $|V(G)|$. The **size** of a graph is the number of edges $|E(G)|$.

Definition 2.2 (Neighbourhood, degree). If G is a graph and $v \in V(G)$, then the **neighbourhood** (or *nbhd*) of v is the set

$$\Gamma(v) = \{u \in V(G) \mid uv \in E(G)\}.$$

The **degree** of a vertex $v \in V$ is the size of its neighbourhood: $d(v) = |\Gamma(v)|$.

Remark 2.3. Beware of other different definitions of graphs! All the graphs in this course will be **simple**, **loopless**, and **undirected**. To be precise, a graph is **simple** if it cannot have multiple edges between two vertices; a graph is **loopless** if every edge contains exactly two different vertices; a graph is **undirected** if its edges are 2-sets such as $\{u, v\}$ rather than ordered pairs such as (u, v) or (v, u). Note that our definition of a graph as $G = (V, E)$ with $E \subseteq \binom{V}{2}$ implies immediately that our graphs are simple, loopless and undirected.

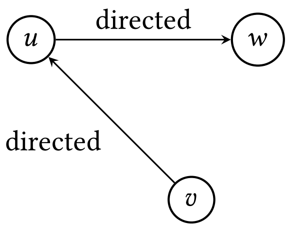

Figure 1. *A directed graph*

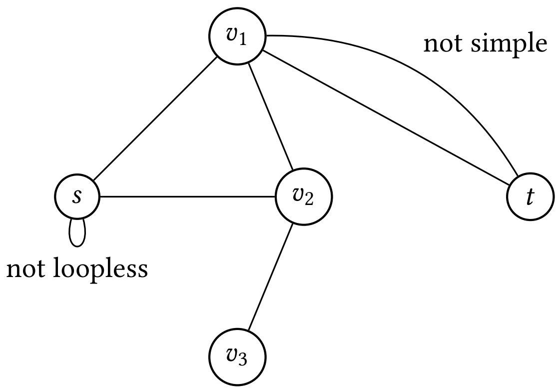

Figure 2. *An undirected graph which is not loopless and not simple*

Remark 2.4. If $e = \{u, v\}$ is an edge in a graph G then we will often write this as $e = uv$. Note that this does not imply that the edge has a direction (our graphs are undirected).

Definition 2.5 (Incident). If $e = uv$ is an edge, then e is **incident** to u and v.

Definition 2.6 (r-regular). A graph G is r**-regular** if $d(v) = r$ for all $v \in V(G)$.

Definition 2.7 (Degree sequence). The **degree sequence** of a graph $G = (V, E)$ is the tuple

$$(d(v_1), d(v_2), \ldots, d(v_n)),$$

where $V = \{v_1, \ldots, v_n\}$ and $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_n)$.

Lemma 2.8 (Handshake Lemma). *For any graph $G = (V, E)$,*

$$\sum_{v \in V} d(v) = 2|E|.$$

Proof. Omitted. □

Lemma 2.9. *In any graph, the number of vertices of odd degree is even.*

Proof. Omitted. □

2.2 Examples of graphs

Recall that for $n \in \mathbb{N}$ we define $[n] = \{1, 2, \ldots, n\}$.

Definition 2.10 (Complete graph). K_n is the **complete graph** of order $n \geq 1$ where

$$V(K_n) = [n], \quad E(K_n) = \binom{[n]}{2}.$$

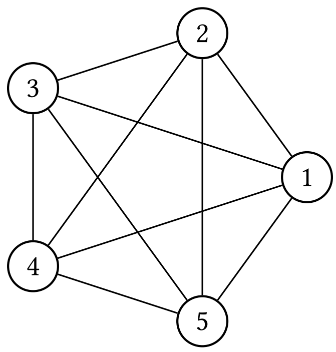

Figure 3. *The complete graph of order* 5, K_5

Definition 2.11 (Empty graph). E_n is the **empty graph** of order $n \geq 1$ where

$$V(E_n) = [n], \quad E(E_n) = \emptyset.$$

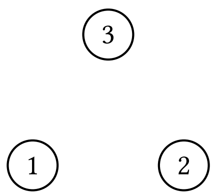

Figure 4. *The empty graph of order 3,* E_3

Definition 2.12 (Cycle). C_n is the **cycle** of length $n \geq 3$ where

$$V(C_n) = [n], \quad E(C_n) = \{\{i, i+1\} \mid i = 1, 2, \ldots, n-1\} \cup \{\{1, n\}\}.$$

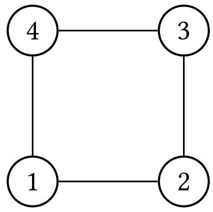

Figure 5. *The cycle of order 4, C_4*

Definition 2.13 (Path). P_n is the **path** of length $n \geq 1$ (with n edges and $n+1$ vertices) where

$$V(P_n) = \{0, 1, 2, \ldots, n\}, \quad E(P_n) = \{\{i-1, i\} \mid i \in [n]\}.$$

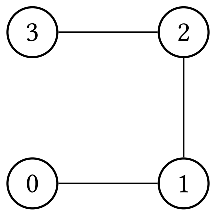

Figure 6. *The path of length 4, P_4*

Definition 2.14 (Complete bipartite graph). $K_{a,b}$ is the **complete bipartite graph** with classes of size a and b ($a, b \geq 1$) where

$$V(K_{a,b}) = \{1, 2, \ldots, a\} \dot\cup \{a+1, a+2, \ldots, a+b\},$$
$$E(K_{a,b}) = \{\{i, j\} \mid 1 \leq i \leq a, a+1 \leq j \leq a+b\}.$$

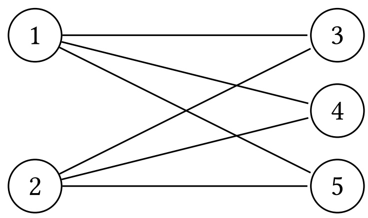

Figure 7. *The complete bipartite graph with classes of size 2 and 3, $K_{2,3}$*

Definition 2.15 (Discrete hypercube). Q_n is the **discrete hypercube** of dimension $n \geq 1$ where

$$V(Q_n) = \{0, 1\}^n, \quad E(Q_n) = \{xy \mid x \text{ and } y \text{ differ in exactly one coordinate}\}.$$

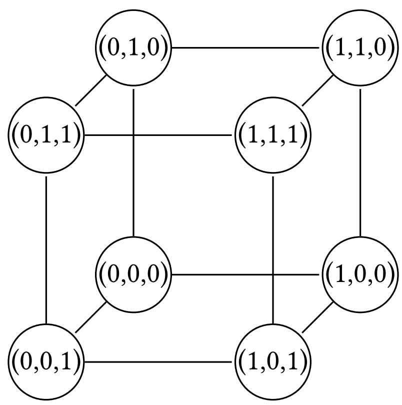

Figure 8. *The discrete hypercube of dimension 3, Q_3*

Remark 2.16. Consider the following graph $G = (V, E)$ where $V = \{2, 3, 4\}$ and $E = \binom{V}{2}$.

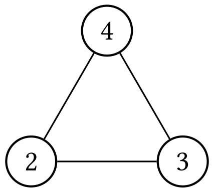

Figure 9. *The 'complete' graph of order 3*

Is G the complete graph of order 3, K_3? Strictly speaking, no since $V \neq \{1, 2, 3\}$. In fact, G is isomorphic to K_3 and we write $G \cong K_3$ (see later). This uniqueness property justifies the word 'the' in our definitions of frequently used graphs. However, for convenience, we often say that G is complete by which we actually mean $G \cong K_3$.

Examples of graphs

Graphs	Notation	Specifics	Vertices	Edges
Complete graph	K_n	$n \geq 1$	$V(K_n) = [n]$	$E(K_n) = \binom{[n]}{2}$
Empty graph	E_n	$n \geq 1$	$V(E_n) = [n]$	$E(E_n) = \emptyset$
Cycle	C_n	$n \geq 3$	$V(C_n) = [n]$	$E(C_n) = \{\{i, i+1\} \mid i = 1, 2, \ldots, n-1\} \cup \{\{1, n\}\}$
Path	P_n	$n \geq 1$	$V(P_n) = \{0, 1, 2, \ldots, n\}$	$E(P_n) = \{\{i-1, i\} \mid i \in [n]\}$
Complete bipartite graph	$K_{a,b}$	classes of size a and b, $(a, b \geq 1)$	$V\left(K_{a,b}\right) = \{1, 2, \ldots, a\} \dot{\cup} \{a+1, a+2, \ldots, a+b\}$	$E\left(K_{a,b}\right) = \{\{i, j\} \mid 1 \leq i \leq a, a+1 \leq j \leq a+b\}$
Discrete hypercube	Q_n	$n \geq 1$	$V(Q_n) = \{0, 1\}^n$	$E(Q_n) = \{xy \mid x$ and y differ in exactly one coordinate $\}$

Graphs	Order	Size	Bipartite	Longest path
Complete graph	n	$\binom{n}{2}$	$n = 1, 2$	$n - 1$
Empty graph	n	0	$\forall n$	0
Cycle	n	n	n is even	$n - 1$
Path	$n + 1$	n	$\forall n$ (no odd cycle)	n
Complete bipartite graph	$a + b$	ab	N/A	$2b - 1$ if $a = b$ $2b$ if $a > b$
Discrete hypercube	2^n	$n2^{n-1}$	$\forall n$	$2^n - 1$

2.3 Subgraphs and isomorphisms

Definition 2.17 (Subgraph). If G and H are graphs satisfying $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a **subgraph** of G.

Definition 2.18 (Induced subgraph). A subgraph H of G is an **induced subgraph** of G if $E(H) = E(G) \cap \binom{V(H)}{2}$. If $G = (V, E)$ is a graph and $A \subseteq V$, then $G[A]$ is the subgraph induced by A : its vertex set is $V(G[A]) = A$ and edge set is $E(G[A]) = E(G) \cap \binom{A}{2}$.

Definition 2.19 (Isomorphic, copy). Graphs G and H are **isomorphic** iff there is a bijection $f : V(G) \to V(H)$ such that $vw \in E(G) \iff f(v)f(w) \in E(H)$. If G and H are isomorphic we denote this by $G \cong H$. G contains a **copy** of H if G has a subgraph isomorphic to H.

2.4 Walks, paths, and connectedness

Definition 2.20 (Walk, closed). A **walk** in G is a sequence of vertices (not necessarily distinct) $v_0v_1\cdots v_t$ such that $v_{i-1}v_i \in E$ for all $1 \le i \le t$. A walk is **closed** if $v_0 = v_t$.

Definition 2.21 (Tour). A walk in which no edge is used more than once is a **tour**.

Remark 2.22. A tour can have repeated vertices but cannot have repeated edges. A path cannot have repeated vertices and hence cannot have repeated edges either.

Lemma 2.23. *Define the relation $\sim$ on $V(G)$ by $v \sim w$ iff there is a walk from v to w in G. Then $\sim$ is an equivalence relation.*

Proof. Omitted. □

Definition 2.24 (Component, connected). Let $V = V_1 \cup V_2 \cup \cdots \cup V_k$ be the partition of V induced by $\sim$. We call the equivalence classes V_i **components**. G is **connected** if it consists of a single component.

Definition 2.25 (Path in a graph, x-y path). A **path in a graph** G is a subgraph isomorphic to P_t for some $t \ge 0$. So it consists of a sequence of distinct vertices $v_0v_1\cdots v_t$ such that $v_{i-1}v_i$ is an edge for $1 \le i \le t$. If $x, y \in V(G)$, then an x-y **path** in G is a path that starts at x and ends at y.

Lemma 2.26. *There is an x-y path in G iff there is a walk from x to y in G.*

Proof. Omitted. □

Lemma 2.27. *Let $P = x_1x_2\ldots x_t$ be a path in a graph G. If P is a shortest x_1-x_t path in G, then $x_1x_2\cdots x_i$ and $x_ix_{i+1}\cdots x_t$ are shortest x_1-x_i and x_i-x_t paths in G for each $1 \le i \le t$.*

Proof. Omitted. □

2.5 Euler circuits

Definition 2.28 (Euler circuit). An **Euler circuit** in a graph G is a closed tour $v_0v_1\cdots v_tv_0$ containing all edges of G.

Remark 2.29. Since an Euler circuit contains all edges, it also contains all vertices. The vertices may be repeated but each edge is used exactly once.

Theorem 2.30 (Euler 1736). *A graph G has an Euler circuit iff G is connected and every vertex has even degree.*

Proof. Omitted. □

2.6 Bipartite graphs

Definition 2.31 (Bipartite graph). A graph G is **bipartite** if

$$V(G) = A\dot{\cup}B \text{ and } E(G) = \{ab \mid a \in A, b \in B\}.$$

We say that A, B is a **bipartition** and sometimes write $G = (A, B; E)$ to emphasise this.

Theorem 2.32. *A graph is bipartite iff it contains no odd cycle.*

Proof. Omitted. □

2.7 Graph colouring

Definition 2.33 (Independent set). Let G be a graph. $A \subseteq V(G)$ is an **independent set** if there are no edges with both endpoints in A.

Definition 2.34 (k-colouring). Let $k \in \mathbb{N}$. A k-**colouring** of a graph G is a function $c : V(G) \longrightarrow [k]$ such that if $vw \in E$, then $c(v) \neq c(w)$.

Definition 2.35 (k-colourable). A graph G is k-**colourable** if G has a k-colouring.

Remark 2.36. It follows that if a graph G is k-colourable, then G is also $k + 1$-colourable. Note that a graph is bipartite iff it is 2-colourable.

Definition 2.37 (k-partite graph). A graph G is k-**partite** if there is a partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ of $V(G)$ into independent sets.

Remark 2.38. Note that a graph is k-partite iff it is k-colourable.

Definition 2.39 (Chromatic number). The **chromatic number** of a graph G is the number

$$\chi(G) = \min\{k \geq 1 \mid G \text{ is } k\text{-colourable }\}.$$

Remark 2.40. A useful fact: If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Note that for any $t \in \mathbb{N}$ we have $\chi(K_t) = t$, $\chi(C_{2t}) = 2$ and $\chi(C_{2t+1}) = 3$.

Definition 2.41 (Maximum degree). The **maximum degree** of a graph G is the number

$$\Delta(G) = \max\{d(v) \mid v \in V(G)\}.$$

Theorem 2.42. *If G is a graph, then*

$$\chi(G) \leq \Delta(G) + 1$$

Proof. Omitted. □

3 The probabilistic method

3.1 Basics

Definition 3.1 (Probability space). A **probability space** is a pair $(\Omega, \mathbf{P})$, where Ω is a (finite) set of elementary events (e.g. {Heads, Tails} or $\{1,2,3,4,5,6\}$) and $\mathbf{P} : \Omega \to [0,1]$ is a function such that $\sum_{\omega\in\Omega} \mathbf{P}[\omega] = 1$.

Definition 3.2 (Event). Any subset $A \subseteq \Omega$ is an **event** and we define its probability to be $\mathbf{P}[A] = \sum_{\omega\in A} \mathbf{P}[\omega]$.

Theorem 3.3 (The Probabilistic Method). *If $(\Omega, \mathbf{P})$ is a probability space and $A \subseteq \Omega$ is an event satisfying $\mathbf{P}[A] > 0$, then $A \neq \emptyset$.*

Proof. Omitted. □

Note that $A \cup B$ and $A \cap B$ denote the events " A or B " and " A and B " respectively. Another very simple but useful result is the probabilistic union bound.

Lemma 3.4 (The union bound). *If $(\Omega, \mathbf{P})$ is a finite probability space and we have events $A_1, \ldots, A_n \subseteq \Omega$, then*

$$\mathbf{P}\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} \mathbf{P}\left[A_i\right].$$

Proof. Omitted. □

Definition 3.5 (Random variable). A **random variable** is a function $X : \Omega \to \mathbb{R}$. We say a random variable is **non-negative** if for all $\omega \in \Omega$, we have $X(\omega) \geq 0$.

Example 3.6. If our probability space is $(\{1,2,3,4,5,6\}, \mathbf{P}_U)$, where $\mathbf{P}_U[\omega] = 1/6$ for all $\omega \in [6]$ (i.e. the space associated with a single fair roll of a die), then we could define the random variables X_1 and X_2 by

$$X_1(\omega) = \begin{cases} 1, & \omega = 1,3,5 \\ 0, & \text{otherwise} \end{cases}$$

and

$$X_2(\omega) = \begin{cases} 1, & \omega \geq 4 \\ 0, & \text{otherwise.} \end{cases}$$

Note that both of these random variables are examples of **indicator random variables**. More generally the indicator random variable of an event $A \subseteq \Omega$ is

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \text{otherwise.} \end{cases}$$

So X_1 is the indicator random variable of "the die roll is odd", while X_2 is the indicator random variable of "the die roll is at least 4 ".

Definition 3.7 (Expectation). The **expectation** of a random variable is simply its average value:

$$\mathbf{E}[X] = \sum_{\omega\in\Omega} X(\omega)\mathbf{P}[\omega] = \sum_{a\in\mathbb{R}} a\mathbf{P}[X = a].$$

Theorem 3.8 (The first moment method). *Let $(\Omega, \mathbf{P})$ be a finite probability space. If X is a random variable on*

$(\Omega, \mathbf{P})$, *then there exist* $\omega_1, \omega_2 \in \Omega$ *such that*

$$X(\omega_1) \leq \mathbf{E}[X] \leq X(\omega_2).$$

Proof. Omitted. □

Lemma 3.9 (Linearity of Expectation). *If* $X_1, X_2, \ldots, X_n$ *are random variables on* $(\Omega, \mathbf{P})$, *then*

$$\mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}[X_i].$$

Proof. Omitted. □

Note that linearity of expectation has nothing to do with independence (which we will define now).

Definition 3.10 (Independence of events, random variables). Let $(\Omega, \mathbf{P})$ be a probability space. Events $A_1, \ldots, A_n \subset \Omega$ are **independent** if for any $1 \leq k \leq n$ of the events, the probability they all hold is the product of their probabilities i.e. for any $1 \leq k \leq n$ and any $\{m_1, \ldots, m_k\} \in \binom{[n]}{k}$

$$\mathbf{P}\left[\bigcap_{i=1}^{k} A_{m_i}\right] = \prod_{i=1}^{k} \mathbf{P}\left[A_{m_i}\right].$$

Random variables X, Y on the same probability space are **independent** if for all $a, b \in \mathbb{R}$, the events $X = a$ and $Y = b$ are independent.

Our first simple application of probability in graph theory is the following result.

Proposition 3.11. *If* G *is a graph of order* n *and size* e, *then* G *contains a bipartite subgraph with at least* $\lceil e/2 \rceil$ *edges.*

Proof. Omitted. □

3.2 Random graphs

The probability space for graphs that we will consider is $\mathscr{G}(n, p)$: the space of Erdös-Renyi random graphs. The underlying set of outcomes is the set of all labelled graphs of order n :

$$\Omega = \left\{G \mid V(G) = [n], E(G) \subseteq \binom{[n]}{2}\right\}.$$

For a graph $H \in \Omega$, the probability of the outcome H which is defined to be $\mathbf{P}[H]$ is the probability that the following random process produces the graph H:

1. Start with $H = E_n$ the empty graph with vertex set $[n]$ and no edges;
2. For each pair of vertices $ij \in \binom{[n]}{2}$, toss a biased coin C_{ij} that has probability p of being 'Heads' and $1 - p$ of being 'Tails'. If C_{ij} is 'Heads' then insert the edge ij, otherwise do not insert the edge ij. All coin tosses are independent.

Note that unless $p \in \{0, 1\}$, every possible graph $H \in \Omega$ has non-zero probability of occuring.

3.3 Large girth and large chromatic number

Definition 3.12 (Girth). The **girth** of a graph G is the length of the shortest cycle in G. We denote this by $g(G)$. If G contains no cycles, then we define $g(G) = +\infty$.

Definition 3.13 (Independence number). The **independence number** of a graph G is the number

$$\alpha(G) = \max\{|A| \mid A \subseteq V(G) \text{ is an independent set }\}.$$

Theorem 3.14 (Erdös 1959). *For all $k, l \geq 3$, there exists a graph G with $\chi(G) \geq k$ and $g(G) \geq l$.*

Proof. Omitted. □

Lemma 3.15. *For any graph G of order n, we have*

$$\chi(G) \geq \frac{n}{\alpha(G)}.$$

Proof. Omitted. □

Lemma 3.16. *Let $G \in \mathscr{G}(n, p)$ and let X_t be the number of t-cycles in G. Then*

$$\mathbf{E}\left[X_t\right] = \frac{n(n-1)(n-2)\cdots(n-t+1)}{2t} p^t.$$

Proof. Omitted. □

Lemma 3.17 (Markov's inequality). *If X is a non-negative random variable and $\lambda > 0$, then*

$$\mathbf{P}[X \geq \lambda] \leq \frac{\mathbf{E}[X]}{\lambda}.$$

Proof. Omitted. □

4 Extremal graph theory

4.1 Hamilton cycles

Definition 4.1 (Hamilton cycle). A **Hamilton cycle** in a graph G is a cycle containing all the vertices of G.

Note that this is a rather different object to an Euler circuit (a closed tour containing all edges of a given graph). Whereas we can view an Euler circuit as a sightseeing tour of a city which must pass along each road exactly once, a Hamilton cycle can be seen as the itinerary of a travelling salesman who wishes to visit every city exactly once, starting and finishing at home.

The question of whether a given graph G contains an Euler circuit has, as we saw a simple characterisation: G contains an Euler circuit iff it is connected and all vertices have even degree. The corresponding question for Hamilton cycles has no such easy answer. (Indeed for those of you who know any computational complexity theory the problem of deciding whether a given graph contains a Hamilton cycle is **NP**-complete. Roughly speaking this means that it is very unlikely that there is any efficient method for deciding if an arbitrary large graph contains a Hamilton cycle.) We will instead consider some sufficient conditions for the existence of Hamilton cycles.

Remark 4.2. A Hamilton cycle is generally not an Euler circuit since it does not necessarily contain all edges. An Euler circuit is generally not a Hamilton cycle since it is not necessarily even a cycle.

Definition 4.3 (Minimum degree). The **minimum degree** of a graph G is the number

$$\delta(G) = \min\{d(v) \mid v \in V(G)\}.$$

Definition 4.4 (Adjacent). Two vertices $u, v \in V(G)$ are **adjacent** if $uv \in E(G)$ otherwise they are **non-adjacent**.

Theorem 4.5 (Dirac 1952). *If G is a graph of order $n \geq 3$ and $\delta(G) \geq n/2$, then G contains a Hamilton cycle.*

Proof. Omitted. □

Note that Dirac's theorem follows immediately from the following result.

Theorem 4.6 (Ore 1960). *If G is a graph of order $n \geq 3$ and $d(u) + d(v) \geq n$ for every pair of non-adjacent vertices $u, v \in V(G)$, then G is Hamiltonian.*

Proof. Omitted. □

4.2 Forbidden subgraphs: Mantel's theorem

Definition 4.7 (H-free). Let G and H be graphs. G is H-**free** if G does not contain a copy of H.

Definition 4.8 (Extremal number, Turán number). The **extremal number** (or **Turán number**) of H is the number

$$\mathrm{ex}(n, H) = \max\{|E(G)| : G = (V, E), |V| = n \text{ and } G \text{ is } H\text{-free}\}.$$

The question of determining the value of $\mathrm{ex}(n, H)$ for a fixed graph H is called the Turán problem for H.

Solving the Turán problem for H really requires us to achieve two objectives. Suppose we want to show $\mathrm{ex}(n, H) = k$:

(i) $\mathrm{ex}(n, H) \geq k$: Find a graph G of order n and size k such that it is H-free;

(ii) $\mathrm{ex}(n, H) \leq k$: If G is of order n and H-free, then $|E(G)| \leq k$.

The following result will help us to achieve $\geq$ in many cases.

Lemma 4.9. *If G and H are graphs with $\chi(H) > \chi(G)$, then G is H-free.*

Proof. Omitted. □

Theorem 4.10 (Mantel 1903). *If* $n \geq 1$*, then* $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$.

Proof. Omitted. □

4.3 Forbidden subgraphs: Turán's theorem

Given (Mantel 1903) and (4.9), an obvious candidate for a K_{r+1}-free graph that has the most edges (i.e. that has size ex (n, K_{r+1})) is a graph with chromatic number r and as many edges as possible subject to this constraint.

Definition 4.11 (Complete r-partite graph). A graph $G = (V, E)$ is a **complete r-partite graph** if there is a partition $V = V_1 \cup V_2 \cup \cdots \cup V_r$, each V_i is an independent set and

$$E(G) = \left\{vw \mid v \in V_i, w \in V_j, \text{for some } 1 \leq i \neq j \leq r\right\}.$$

(i.e. all edges between distinct vertex classes are present.)

Clearly, among all r-partite graphs of order n the number of edges will be maximised by a complete r-partite graph (since if an r-partite graph is not complete then we can add an edge while still maintaining its chromatic number as r). But how should the n vertices be shared among the r vertex classes? For $r = 2$ we can easily check that if the two vertex classes have a and $n - a$ vertices then the number of edges is $a(n - a)$ and the is easily seen to be maximised when $a = \lfloor n/2 \rfloor$. But for $r > 2$ this problem is a little less straightforward.
In fact, it turns out that taking the r classes to be as equal as possible in size will achieve the desired result.
Fact: Turán graphs have maximal number of edges among r-partite graphs with vertex set $[n]$.

Definition 4.12 (Turán graph). Let $n \geq r \geq 2$ be integers. The **Turán graph** $T_r(n)$ is the complete r-partite graph with the vertex set $[n]$ and vertex classes as equal as possible in size.

Remark 4.13. Note that this defines a unique (upto isomorphism) r-partite graph of order n, with b vertex classes each containing $\lfloor n/r \rfloor$ vertices and $r - b$ vertex classes each containing $\lceil n/r \rceil$ vertices, where b satisfies $n = b\lfloor n/r \rfloor + (r - b)\lceil n/r \rceil$. **Denote the number of edges in $T_r(n)$ by $t_r(n)$.**

Note that if we really wished to we could give an explicit formula for $t_r(n)$ but it would not in general be very useful so we do not bother!
The next result tells us that $T_r(n)$ is a very plausible candidate for solving the Turán problem for K_{r+1} in the sense that the converse of the above **fact** also holds.

Lemma 4.14. *Let G be an r-partite graphs with n vertices and maximal edges. Then G is isomorphic to* $T_r(n)$*. Moreover*

$$t_r(n) = t_r(n - r) + (r - 1)(n - r) + \binom{r}{2}.$$

Proof. Omitted. □

Theorem 4.15 (Turán 1, 1941). *If* $2 \leq r \leq n$ *are integers and G is a* K_{r+1}*-free graph of order n, then* $|E(G)| \leq t_r(n)$.

Proof. Omitted. □

Theorem 4.16 (Turán 2, 1941). *If* $2 \leq r \leq n$ *are integers and G is a* K_{r+1}*-free graph of order n with* $\mathrm{ex}(n, K_{r+1})$ *edges, then G is isomorphic to* $T_r(n)$.

Proof. Omitted. □

4.4 Digression: double counting

Theorem 4.17 (Double counting principle). *If $G = (A, B; E)$ is a bipartite graph, then*

$$\sum_{a \in A} d(a) = \sum_{b \in B} d(b).$$

Proof. Omitted. □

We have already seen examples of such arguments but we have not explicitly used this bipartite graph formulation (mainly because it would have made our arguments more complicated).
For example, the Handshake Lemma says: if $G = (V, E)$ is a graph then $\sum_{v \in V} d(v) = 2|E|$. This can be proved using an explicit double counting argument as follows. Let $G = (V, E)$ be a graph. Now define a bipartite graph $H = (A, B; F)$ where $A = V, B = E$ and the edges in H are

$$F = \{ve \mid v \in V, e \in E \text{ and } v \in e\}.$$

So H is a bipartite graph with edge set F. Moreover, using subscripts to denote degrees in the two different graphs, we have

$$\sum_{a \in A} d_H(a) = \sum_{v \in V} d_G(v)$$

while

$$\sum_{b \in B} d_H(b) = \sum_{e \in E} \#\{v \mid v \in e\} = \sum_{e \in E} 2 = 2|E|.$$

The double counting principle then tells us that these two expressions are equal.

4.5 Asymptotics: Turán density

Definition 4.18 (Turán density). The **Turán density** of a graph F is the number

$$\pi(F) = \lim_{n \to \infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{2}}.$$

Lemma 4.19. *For a graph F, $\pi(F)$ is well defined. If $r \geq 2$, then $\pi(K_{r+1}) = 1 - 1/r$.*

Proof. Omitted. □

4.6 Bipartite forbidden subgraphs

Turán's theorem gives us a full answer to the Turán problem for complete graphs, but what can we say for bipartite graphs? It is easy to compute $\mathrm{ex}(n, K_{1,t})$ directly for any $t \geq 1$, but in general the problem is hard and we settle for upper bounds.

Remark 4.20. Let G be a $K_{1,t}$-free graph of order n. For each $v \in G$, v can be adjacent to at most $t - 1$ vertices:

$$\mathrm{ex}\left(n, K_{1,t}\right) = \frac{(t-1)n}{2}.$$

The result is divided by 2 since each edge contains 2 vertices.

Theorem 4.21 (Kővári-Sós-Turán 1954). *If $n \geq r \geq s \geq 2$, then*

$$\mathrm{ex}(n, K_{r,s}) \leq \frac{1}{2}(r-1)^{1/s} n^{2-1/s} + \frac{1}{2}(s-1)n.$$

In particular, $\mathrm{ex}(n, K_{r,s}) = \mathcal{O}\left(n^{2-1/s}\right)$ *and* $\pi\left(K_{r,s}\right) = 0$.

Proof. Omitted. □

Corollary 4.22 (Erdős 1946). *If $X \subset \mathbb{R}^2$ and $|X| = n$, then at most $\frac{n^{3/2}}{\sqrt{2}} + \frac{n}{2}$ pairs of points in X are at unit distance.*

Proof. Omitted. □

4.7 Erdös-Stone: the fundamental theorem of extremal graph theory

Turán's theorem implies that for $r \geq 3$

$$\pi(K_r) = 1 - \frac{1}{r-1} = 1 - \frac{1}{\chi(K_r) - 1}.$$

So in these cases the Turán density is determined by the chromatic number. Moreover, this also holds for complete bipartite graphs by the Kövári-SósTurán theorem (4.21) since

$$\pi(K_{r,s}) = 0 = 1 - \frac{1}{\chi(K_{r,s}) - 1}.$$

In fact, this holds in general and so allows us determine the Turán density of any graph in terms of its chromatic number.

Theorem 4.23 (Erdős-Stone 1946). *If H is a graph with chromatic number $\chi(H) = r$, then*

$$\pi(H) = 1 - \frac{1}{r-1}.$$

Proof of (4.23) $\implies$. Omitted. □

Proof of (4.23) $\impliedby$. Omitted. □

Lemma 4.24. *Let $0 < c, \epsilon < 1$ and $n > 2(1 + 1/c)/\epsilon$. If G is a graph of order n with at least $(c + \epsilon)\binom{n}{2}$ edges, then G contains a subgraph G' of order $n' \geq \epsilon^{1/2}n$ with minimum degree $\delta(G') \geq cn'$.*

Proof. Omitted. □

Theorem 4.25. *Let $r \geq 2, t \geq 1$ and $0 < \epsilon < 1/r$. There exists $n_0(r, t, \epsilon)$ such that if G has $n \geq n_0$ vertices and minimum degree*

$$\delta(G) \geq \left(1 - \frac{1}{r-1} + \epsilon\right) n,$$

then G contains a copy of $K_r(t)$.

Proof. Omitted. □

4.8 Stability

If a K_3-free graph of order n has "almost" $\text{ex}(n, K_3) = \lfloor n^2/4 \rfloor$ edges must it look like the Turán graph $T_2(n)$?

Theorem 4.26 (Füredi 2010). *If G is a K_{r+1}-free graph of order n with at least $\text{ex}(n, K_{r+1}) - t$ edges, for some $t \geq 0$, then there exists $H \subseteq G$ such that $|E(H)| \geq |E(G)| - t$ and $\chi(H) \leq r$.*

Proof. Omitted. □

5 Families of sets: chains, antichains, and intersection problems

5.1 Chains and antichains

Let $0 \le k \le n$ be integers and recall that $\mathscr{P}([n])$ denotes the power set of $[n]$:

$$\mathscr{P}([n]) = \{A \mid A \subseteq [n]\}$$

while the family of k-subsets of $[n]$ is

$$\binom{[n]}{k} = \{A \subseteq [n] \mid |A| = k\}.$$

Definition 5.1 (Chain). A family of sets $\mathscr{C}$ is a **chain** if $\forall A, B \in \mathscr{C} A \subseteq B$ or $B \subseteq A$.

Remark 5.2. A chain is a family of sets $\mathscr{C}$ that can be linearly ordered under inclusion i.e. $\mathscr{C} = \{C_1, C_2, \ldots, C_t\}$ with $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_t$.

Definition 5.3 (Antichain). A family of sets $\mathscr{A}$ is an **antichain** if $\forall A, B \in \mathscr{A}, A \subseteq B \implies A = B$.

Remark 5.4. An antichain is a family of sets $\mathscr{A}$ that are incomparable under inclusion i.e. if $A, B \in \mathscr{A}$ and $A \neq B$ then $A \not\subseteq B$ and $B \not\subseteq A$.

The first question we will explore in this section is: if $\mathscr{A} \subseteq \mathscr{P}([n])$ is a chain or antichain how large can $\mathscr{A}$ be?
For chains this question is trivial (stop and figure out the answer for yourself if it isn't immedidately obvious). For antichains the answer requires some work.
Both versions of this question require the following simple fact.

Lemma 5.5. *If $\mathscr{A}$ is an antichain and $\mathscr{C}$ is a chain, then*

$$|\mathscr{A} \cap \mathscr{C}| \le 1.$$

Proof. Omitted. □

How large can a chain $\mathscr{C} \subseteq \mathscr{P}([n])$ be?

Proposition 5.6. *If $\mathscr{C} \subseteq \mathscr{P}([n])$ is a chain, then $|\mathscr{C}| \le n + 1$.*

Proof. Omitted. □

A little thought tells us that an obvious candidate for the largest antichain in $\mathscr{P}([n])$ is the "middle layer": $\binom{[n]}{\lfloor n/2 \rfloor}$. This guess turns out to be correct, and we can prove it using the same basic idea as Proposition (5.6) by finding a suitable partition of $\mathscr{P}([n])$ into chains.

Theorem 5.7 (Sperner 1928). *If $\mathscr{A} \subseteq \mathscr{P}([n])$ is an antichain, then*

$$|\mathscr{A}| \le \binom{n}{\lfloor n/2 \rfloor}.$$

By Lemma (5.5), this result will follow if we can show that $\mathscr{P}([n])$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains. In fact, we will prove a slightly stronger result.

Definition 5.8 (Symmetric). A chain $\mathscr{C} \subseteq \mathscr{P}([n])$ is **symmetric** if $\mathscr{C} = \{C_1, \ldots, C_k\}$ with $|C_{i+1}| = |C_i| + 1$ for all $1 \le i \le k - 1$ and $|C_1| + |C_k| = n$.

Lemma 5.9. *$\mathscr{P}([n])$ can be partitioned into symmetric chains.*

Proof. Omitted. □

Proof of (5.7). Omitted. □

5.2 LYM-inequality

Theorem 5.10. *(Lubell, Yamamoto, Meshalkin 1954). If $\mathscr{A} \subseteq \mathscr{P}([n])$ is an antichain, then*

$$\sum_{A\in\mathscr{A}} \frac{1}{\binom{n}{|A|}} \leq 1.$$

Remark 5.11. Note that 2 terms $\binom{n}{|A_1|}, \binom{n}{|A_2|}$ in this summation is the same if $|A_1| = |A_2|$. For $0 \leq k \leq n$, let $a_k = |\mathscr{A} \cap \binom{[n]}{k}|$ denote the number of size k sets in $\mathscr{A}$, then equivalenly,

$$\left(\sum_{A\in\mathscr{A}} \frac{1}{\binom{n}{|A|}} =\right) \quad \sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} \leq 1.$$

We are simply gathering same terms.

Remark 5.12. If one sums the proportion of each layer contained in $\mathscr{A}$ over all of the layers, the sum of that proportion ≤ 1.

Proof by counting. Omitted. □

Proof by the probabilistic method. Omitted. □

5.3 Intersecting families

Definition 5.13 (Intersecting family). A family of sets $\mathscr{A}$ is **intersecting** if $\forall A, B \in \mathscr{A}, A \cap B \neq \emptyset$.

How large can an intersecting family $\mathscr{A} \subseteq \mathscr{P}([n])$ be?

Proposition 5.14. *If $\mathscr{A} \subseteq \mathscr{P}([n])$ is intersecting, then $|\mathscr{A}| \leq 2^{n-1}$.*

Remark 5.15. Note that 2^{n-1} is tight since the family $\mathscr{A} = \{A \in \mathscr{P}([n]) \mid 1 \in A\}$ has size 2^{n-1}.

Proof. Omitted. □

If $\mathscr{A} \subseteq \binom{[n]}{k}$ is intersecting, how large can it be?

- If $n < 2k$, then $\binom{[n]}{k}$ is intersecting. This is because that we need at least $2k$ elements to have 2 disjoint sets of size k. Hence,
$$|\mathscr{A}| \leq \binom{n}{k}.$$
- If $n = 2k$, for $\mathscr{A}$ to be intersecting, one needs $A \in \mathscr{A} \implies [n] \setminus A \notin \mathscr{A}$. In this case,
$$|\mathscr{A}| \leq \frac{1}{2}\binom{2k}{k} = \frac{k}{n}\binom{n}{k}.$$
- If $n > 2k$, which is the really interesting case, one large intersecting family is
$$\mathscr{A}^* = \{A \in \binom{[n]}{k} \mid 1 \in A\} \text{ and } |\mathscr{A}^*| = \binom{n-1}{k-1}.$$

  Thus, the next theorem tells us that one cannot do any better than this (i.e. there is no larger upper bounds).

Theorem 5.16 (Erdős-Ko-Rado 1961). *If $\mathscr{A} \subseteq \binom{[n]}{k}$ is intersecting and $n \geq 2k$, then*

$$|\mathscr{A}| \leq \binom{n-1}{k-1}.$$

Remark 5.17. Note that $\binom{n-1}{k-1} = \frac{k}{n}\binom{n}{k}$, which is the bound we find for $n = 2k$ case.

Proof by using cyclic permutations due to G.O.H. Katona 1972. Omitted. □

5.4 Compressions (not lectured 2023 and non-examinable)

5.5 The linear algebra method

Definition 5.18 (Linearly independent). A set of vectors $\{v_1, \dots, v_t\}$ in a vector space V over a field $\mathbb{F}$ is **linearly independent** if

$$\sum_{i=1}^{t} \lambda_i v_i = 0 \text{ with } \lambda_1, \dots, \lambda_t \in \mathbb{F} \implies \lambda_1 = \lambda_2 = \cdots = \lambda_t = 0.$$

Lemma 5.19 (The linear algebra bound). *If $v_1, \dots, v_t$ are linearly independent vectors in a vector space of dimension d, then $t \leq d$.*

Proof. This is part of the Steinitz Exchange Lemma (see 1st Year Algebra 1). □

Definition 5.20 (Inner product space, pre-Hilbert space). Let X be any vector space over $\mathbb{F}$. An **inner product** on X is a function $(\cdot,\cdot) : X \times X \to \mathbb{F}$ (often called **Hermitian** if $\mathbb{F} = \mathbb{C}$) satisfying $\forall x, y \in X\ \forall \alpha \in \mathbb{F}$

- Linearity:
$$(x + y, z) = (x, z) + (y, z);$$
- Homogeneity:
$$(\alpha x, y) = \alpha(x, y);$$
- Conjugate symmetry:
$$(x, y) = \overline{(y, x)};$$
- Non-degeneracy (positive definite):
$$\begin{cases} (x, x) & \geq 0, \\ (x, x) & = 0 \Leftrightarrow x = 0. \end{cases}$$

The pair $(X, (\cdot,\cdot))$ is an **inner product space** (or a **pre-Hilbert space**).

Remark 5.21. Note that $(0, x) = (0, x) + (0, x) = 0$.

Definition 5.22 (Orthogonal). A set of vectors $\{v_1, \dots, v_t\}$ in an inner product space is **orthogonal** if $\forall i \neq j$, $\langle v_i, v_j \rangle = 0$.

Remark 5.23. It is easy to check that any orthogonal set of vectors is linearly independent hence the bound in Lemma (5.19) applies to any orthogonal set of vectors. Let $\{v_1, v_2, \dots, v_n\}$ be a set of **non-zero** orthogonal vectors. Let $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{F}$ such that

$$\sum_{k=1}^{n} \lambda_k v_k = 0,$$

then

$$\lambda_j \cancelto{\neq 0}{\|v_j\|^2} = \left\langle \sum_{k=1}^{n} \lambda_k v_k, v_j \right\rangle = 0, \quad \forall j = 1, 2, \ldots, n$$

by linearity of the first argument of the inner product.

We start with a simple example to illustrate the method. The basic idea is to associate vectors with sets from a given family and then show that the vectors we obtain are linearly independent and hence prove an upper bound on the size of the family.

Theorem 5.24. *If $\mathcal{A} = \{A_1, \ldots, A_m\} \subseteq \mathcal{P}([n])$ is a family of sets satisfying:*

(i) $|A_i|$ is odd for all $1 \le i \le m$;

(ii) $|A_i \cap A_j|$ is even for all $i \ne j$,

then $m \le n$.

Proof. Omitted. □

A more interesting application is the following result known as Fisher's Inequality.

Theorem 5.25 (Fisher 1940). *Let $k \ge 1$. If $\mathcal{A} \subseteq \mathcal{P}([n])$ satisfies $|A \cap B| = k$ for every pair of sets $A, B \in \mathcal{A}$ with $A \ne B$, then $|\mathcal{A}| \le n$.*

Proof. Omitted. □

5.6 L-intersecting families

Our final application of the linear algebra method is to a more sophisticated intersection problem. This will require us to work over a vector space of polynomials in several variables.

Recall that $\mathbb{R}[x]$ denotes the **ring of polynomials with real coefficients**. This is

$$\mathbb{R}[x] = \left\{ p(x) = c_0 + c_1 x + \cdots + c_d x^d \mid d \in \mathbb{Z}^+, c_0, \ldots, c_d \in \mathbb{R} \right\}.$$

It is easy to check that this forms a vector space over $\mathbb{R}$ under the obvious operations of addition and scalar multiplication. The zero vector is the zero polynomial.

If $x_1, \ldots, x_n$ are variables, then we can form the obvious generalisation of $\mathbb{R}[x]$, that is, the **ring of multivariate polynomials with real coefficients** $\mathbb{R}[x_1, \ldots, x_n]$. Formally, we define a **monomial** in $x_1, \ldots, x_n$ to be any product of the form $x_{a_1}^{\alpha_1} \cdots x_{a_r}^{\alpha_r}$ where $r \in \mathbb{Z}^+, 1 \le a_1 < a_2 < \cdots < a_r \le n$ and $\alpha_1, \ldots, \alpha_r \in \mathbb{N}$. Note that the empty product is allowed ($r = 0$), and this is defined to be 1. A **multivariate polynomial** is then any finite real linear combination of monomials. The set of all such polynomials is $\mathbb{R}[x_1, \ldots, x_n]$. The **degree of a monomial** $x_{a_1}^{\alpha_1} \cdots x_{a_r}^{\alpha_r}$ is $\sum_{i=1}^{r} \alpha_i$ and the **degree of a non-zero polynomial** $p(x_1, \ldots, x_n)$ is the maximum of the degrees of the monomials it contains.

For example, in $\mathbb{R}[x_1, x_2, x_3, x_4, x_5]$ the polynomial

$$p(x_1, x_2, x_3, x_4, x_5) = -3 + 2x_1 x_3 x_5^2 + 3x_1^3 x_2^2,$$

has degree: $\deg(p) = \max\{0, 4, 5\} = 5$.

Again, it is easy to check that $\mathbb{R}[x_1, \ldots, x_n]$ is a vector space over $\mathbb{R}$.

We will be interested in a special subspace of this space. Let $s \in \mathbb{N}$ and define

$$U(s) = \operatorname{Span}\left\{ x_{a_1} x_{a_2} \cdots x_{a_r} \mid 0 \le r \le s \text{ and } 1 \le a_1 < a_2 \cdots < a_r \le n \right\}.$$

Thus, $U(s)$ is the subspace of $\mathbb{R}[x_1, \ldots, x_n]$ spanned by all monomials of degree at most s with no powers of any variable greater than one. Since any spanning set of vectors contains a linearly independent spanning set (i.e. the basis), we have $\dim(U(s)) \le |U(s)| = \sum_{r=0}^{s} \binom{n}{s}$. This fact is useful when we prove theorem (5.28).

The following lemma is useful for the next theorem.

Lemma 5.26. *If* $q_1, \dots, q_m \in \mathbb{R}[x_1, \dots, x_n]$ *and* $v_1, \dots, v_m \in \mathbb{R}^n$ *satisfy*

(i) *for* $1 \le i \le m$ *we have* $q_i(v_i) \neq 0$;

(ii) *for* $1 \le j < i \le m$ *we have* $q_i(v_j) = 0$,

then $\{q_1, \dots, q_m\}$ *are linearly independent.*

Proof. Omitted. □

Having introduced all of the necessary algebra, we now introduce one more definitino and the combinatorial problem that we wish to consider.

Definition 5.27 (L-intersecting). *Let* $L \subseteq \{0, 1, 2, \dots, n\}$. A family $\mathscr{A} \subseteq \mathscr{P}([n])$ is L-**intersecting** if for each pair of sets $\forall A, B \in \mathscr{A}$ with $A \neq B$, we have $|A \cap B| \in L$.

Theorem 5.28 (Ray-Chaudhuri and Wilson 1975). *If* $\mathscr{A} \subseteq \mathscr{P}([n])$ *is* L *intersecting and* $|L| = s$, *then*

$$|\mathscr{A}| \le \sum_{r=0}^{s} \binom{n}{r}.$$

Proof. Omitted. □

6 Ramsey theory

Ramsey theory is another branch of extremal combinatorics, it has been summarised as saying that "total disorder is impossible."
In Turán-type problems, we consider how dense an object (in our examples this was typically a graph) needs to be to guarantee that it contains a copy of a given sub-object. Ramsey theory instead considers questions of the form "given an object that is partitioned into two (or more) parts, how large must the object be to guarantee that one of the parts contains a particular sub-object".
For example, if we take the set of integers $[N] = \{1, \ldots, N\}$ and partition it into two parts: $[N] = A \dot{\cup} B$, how large must N be to guarantee that one of the parts contains a three term arithmetic progression?
We will consider such questions of Ramsey theory in the integers later, but we start with the Ramsey theory of complete graphs.
We will describe partitions in terms of colourings. So, for example, a partition of the edges of K_n into two parts is described by a red-blue edge-colouring of K_n. Note that such edge-colourings are not graph colourings as considered earlier in this course, there are no constraints on how the edges of K_n are coloured, it is simply a convenient way of describing a partition of the edges of K_n.

Definition 6.1 (Edge-colouring of complete graph). An **edge-colouring** of K_n with k colours $c_1, \ldots, c_k$ is a function $c : E(K_n) \to \{c_1, c_2, \ldots, c_k\}$.

Thus, an edge-colouring of K_n is simply an assignment of colours to the edges of K_n. Given a copy of K_n together with an edge-colouring, we say that K_n is **edge-coloured**. For most of this section we will only consider edge-colourings with two colours which we will take to be red and blue.
Given a red-blue edge-coloured K_n, we say that it contains a red (blue) H if there is a subgraph isomorphic to H with all edges coloured red (blue). A subgraph of an edge-coloured K_n is said to be **monochromatic** if all of its edges have the same colour.
Given an edge coloured K_n and a colour c_i, we define for each $v \in V(K_n)$

$$\Gamma_{c_i}(v) = \{w \in V(K_n) \mid c(vw) = c_i\} \quad \text{and} \quad d_{c_i}(v) = \left|\Gamma_{c_i}(v)\right|.$$

6.1 Ramsey's theorem

Any gathering of six people must contain either three mutual friends or three mutual strangers. This is the first not entirely trivial example of a Ramsey number.

Definition 6.2 (Ramsey number). Let $s, t \geq 2$ be integers. The **Ramsey number** $R(s, t)$ is the smallest integer n such that any red-blue edge-coloured K_n always contains a red K_s or a blue K_t i.e.

$$R(s, t) = \min\{n \in \mathbb{N} \mid \text{any red-blue edge-coloured } K_n \text{ contains a red } K_s \text{ or a blue } K_t\}.$$

Remark 6.3. Let $R_{s,t} = \{n \in \mathbb{N} \mid \text{any red-blue edge-coloured } K_n \text{ contains a red } K_s \text{ or a blue } K_t\}$. Note that it is not obvious that $R(s, t)$ is well-defined since it is possible that $R_{s,t} = \emptyset$. However, it turns out that $R(s, t)$ is well-defined and this fact is known as Ramsey's theorem (6.6). Before stating and proving Ramsey's theorem, we start with some small exact examples.

Proposition 6.4. $R(3, 3) = 6$.

Proof. Omitted. □

Proposition 6.5. $R(3, 4) = 9$.

Proof. Omitted. □

Theorem 6.6 (Ramsey 1930). *For $s, t \geq 2$, there exists $n \in \mathbb{N}$ such that any red-blue edge-coloured K_n contains either* *a red K_s or a blue K_t. Moreover if $R(s,t)$ denotes the smallest such n, then*

$$R(s,t) \leq \binom{s+t-2}{s-1}.$$

Proof. Omitted. □

Proposition 6.7. $R(4,4) = 18$.

Proof. Omitted. □

Theorem 6.8 (Campos, Griffiths, Morris, Sahasrabudhe 2023+). *There exists a real number $\epsilon > 0$ such that*

$$R(s,s) \leq (4-\epsilon)^s.$$

See this for details.

6.2 Ramsey numbers: lower bounds and more colours

What about a lower bound for $R(s,s)$? To show that a given integer n is a lower bound for $R(s,s)$, we need to show that there exists a red-blue edge-colouring of K_n with no monochromatic K_s.

Theorem 6.9 (Erdős 1947). *Let $3 \leq s \leq n$. If $\binom{n}{2}2^{1-\binom{s}{2}} < 1$, then $R(s,s) > n$. In particular, $R(s,s) > 2^{s/2}$.*

Proof. Omitted. □

Definition 6.10 (Multicolour Ramsey number). For $k \geq 1$ and $s_1, \ldots, s_k \geq 2$, define $R_k(s_1, s_2, \ldots, s_k)$ to be the smallest integer n such that for any edge-colouring of K_n with k colours $c_1, \ldots, c_k$, there exists a c_i-coloured copy of K_{s_i} for some $1 \leq i \leq k$ i.e.

$$\begin{aligned} R_k(s_1, s_2, \ldots, s_k) = \min\{n \in \mathbb{N} \mid &\text{ for any edge-colouring of } K_n \text{ with } k \text{ colours } c_1, \ldots, c_k, \\ &\text{there exists a } c_i\text{-coloured copy of } K_{s_i} \text{ for some } 1 \leq i \leq k\}. \end{aligned}$$

If $s_1 = s_2 \cdots = s_k = s$, then we denote this by $R_k(s)$.

Remark 6.11. For example, $R_k(3)$ is the smallest integer n such that whenever the edges of K_n are coloured with k-colours, there exists a monochromatic triangle. As in the case of two colours, we need to check that $R_k(s_1, s_2, \ldots, s_k)$ is in fact well defined.

Theorem 6.12. *For all $k \geq 1$ and $s_1, \ldots, s_k \geq 2$, $R_k(s_1, \ldots, s_k)$ is well defined.*

Proof. Omitted. □

6.3 Ramsey theory in the integers

Theorem 6.13 (Fermat's Last Theorem). *There are no non-trivial integer solutions to $x^n + y^n = z^n$ for any integer $n \geq 3$.*

Proof. The proof of Fermat's Last Theorem is unfortunately slightly too long to fit in these notes. □

Instead we will consider the question of solutions to the Fermat equation modulo a prime p. For example $111^{333} + 222^{333} = 515^{333}$ mod 1051. Our next result tells us that for any fixed n there are always non-trivial solutions modulo any sufficiently large prime.

Theorem 6.14. *For every integer $n \geq 1$ there exists p_n such that for any prime $p \geq p_n$ the congruence*

$$x^n + y^n = z^n \mod p$$

has a non-trivial solution (i.e. a solution with $x, y, z \neq 0 \mod p$).

The key to proving Theorem (6.14) is the following Ramsey type result in the integers known as **Schur's theorem**, which is itself proved using the Ramsey theory of graphs.

Definition 6.15 (Colouring of integer). An **integer-colouring** of $A \subseteq \mathbb{N}$ with k colours $c_1, \ldots, c_k$ is a function $c : A \longrightarrow \{c_1, c_2, \ldots, c_k\}$.

As with the edge-colourings of K_n in the previous section, there are no restrictions on these k-colourings. A k-colouring simply describes a partition of the set A into k parts.

Theorem 6.16 (Schur 1916). *For any $k \geq 1$, there exists an integer $S(k)$ such that for any k-colouring of the integers $\{1, 2, \ldots, S(k)\}$, there exist u, v, w of the same colour such that $u + v = w$.*

Proof. Omitted. □

Lemma 6.17. *If p is a prime, then $\mathbb{Z}_p^*$, the multiplicative group of units* mod p, *is a cyclic group.*

Proof. See 2nd year Algebra 4 notes or Number theory notes. □

Proof of (6.14). Omitted. □

Theorem 6.18. *In any red-blue edge-colouring of $K_{\mathbb{N}}$, there is an infinite set $A \subseteq \mathbb{N}$ such that K_A is monochromatic.*

Proof. Omitted. □

6.4 Van der Waerden's theorem

The last result of our course is the starting point for many more recent deep results in combinatorics and additive number theory, such as the HalesJewett theorem and the Green-Tao theorem on arithmetic progressions in the primes.

Theorem 6.19 (Van der Waerden 1927). *If $k, t \geq 1$, there exists an integer $W(k, t)$ such that whenever $[W(k, t)]$ is k-coloured there is a monochromatic arithmetic progression of length t.*

Proof. Omitted. □

7 Inequalities

Lemma 7.1. *If $n \geq k \geq 1$, then*

$$\frac{(n-k+1)^k}{k!} \leq \binom{n}{k} \leq \frac{n^k}{k!}.$$

Proof. Omitted. □

Definition 7.2 (Convex function). A function $f : (a, b) \to \mathbb{R}$ is **convex** if for all $x, y \in (a, b)$ and $\lambda \in [0, 1]$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

For example $f(x) = x^2$ is convex on $\mathbb{R}$.

Lemma 7.3. *If $f : (a, b) \to \mathbb{R}$ is differentiable and $f'(x)$ is non-decreasing on (a, b), then f is convex. In particular if $f''(x) > 0$, then f is convex.*

Proof. Omitted. □

Definition 7.4. Let $k \geq 1$ be an integer. We extend the " k-th binomial coefficient function" to the real numbers $x \in \mathbb{R}$ as follows:

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!},$$

for $x > k-1$ and $\binom{x}{k} = 0$ for $x \leq k-1$.

Lemma 7.5. *Let $k \geq 1$ be an integer, then $q_k(x) = \binom{x}{k}$ is convex on $\mathbb{R}$.*

Proof. Omitted. □

One inequality to rule them all...

Theorem 7.6 (Jensen's Inequality). *If $\varphi : (a, +\infty) \to \mathbb{R}$ is convex, $\lambda_1, \ldots, \lambda_n \in [0, 1]$ satisfy $\sum_{i=1}^{n} \lambda_i = 1$, and $x_1, \ldots, x_n \in (a, +\infty)$ then*

$$\varphi\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i \varphi(x_i).$$

Proof. Omitted. □

Corollary 7.7 (Simple Cauchy-Schwarz). *If $x_1, \ldots, x_n \in \mathbb{R}$, then*

$$\frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2 \leq \sum_{i=1}^{n} x_i^2.$$

Proof. Omitted. □

Corollary 7.8 (Binomial Coefficient Convexity). *If $x_1, \ldots, x_n \in \mathbb{R}$, then*

$$\binom{\sum_{i=1}^{n} x_i}{k} \leq \frac{1}{n} \sum_{i=1}^{n} \binom{x_i}{k}.$$

Proof. Omitted. □